Continued Fractions

Chao Qin

School of Mathematical Sciences

June 4, 2024

Continued Fractions

Example

Take $a = 43, b = 19$.

$$43 = 2 \times 19 + 5$$

$$19 = 3 \times 5 + 4$$

$$5 = 1 \times 4 + 1$$

$$4 = 4 \times 1 + 0$$

Hence, by Euclids algorithm, the gcd of 43 and 19 is 1.

Continued Fractions

Observe that the quotient at each step of the algorithm has been highlighted. Using these numbers we can present the fraction $\frac{43}{19}$ in the following manner:

$$\frac{43}{19} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}$$

Continued Fractions

Example

$$\frac{225}{157} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}}$$

Continued Fractions

Its convergents are

$$1 = \frac{1}{1}$$

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \frac{1}{2 + \frac{1}{3}} = \frac{10}{7}$$

$$1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4}}} = \frac{43}{30}$$

$$\frac{225}{157} = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}}$$

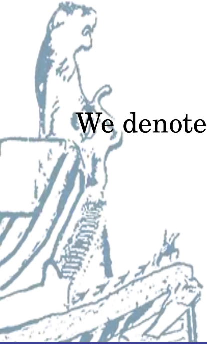
Continued Fractions

A continued fraction continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

We denote the continued fraction displayed above by

$$[a_0, a_1, a_2, \ldots].$$



Continued Fractions

For example,

$$[1,2] = 1 + \frac{1}{2} = \frac{3}{2},$$

$$[3,7,15,1,292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

$$= \frac{103993}{33102} = 3.14159265301190260407\ldots,$$

Continued Fractions

and

$$[2,1,2,1,1,4,1,1,6] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6}}}}}}}}$$

$$= \frac{1264}{465}$$

$$= 2.71827956989247311827956 98\ldots$$

Continued Fractions

Finite Continued Fractions

Definition (Finite Continued Fraction)

A *finite continued fraction* is an expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

where each a_m is a real number and $a_m > 0$ for all $m \geq 1$.

Finite Continued Fractions

Definition (Simple Continued Fraction)

A *simple continued fraction* is a finite or infinite continued fraction in which the a_i are all integers.

Finite Continued Fractions

To get a feeling for continued fractions, observe that

$$\begin{aligned} [a_0] &= a_0, \\ [a_0, a_1] &= a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}, \\ [a_0, a_1, a_2] &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1}. \end{aligned}$$

Also,

$$\begin{aligned} [a_0, a_1, \ldots, a_{n-1}, a_n] &= \left[a_0, a_1, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}\right] \\ &= a_0 + \frac{1}{[a_1, \ldots, a_n]} \\ &= [a_0, [a_1, \ldots, a_n]]. \end{aligned}$$

Finite Continued Fractions

Example

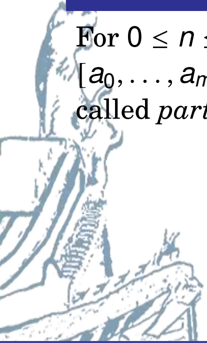
$$[1, 2, 3, 4, 5] = [1, 2, 3, 4, 4, 1]$$

$$\frac{3}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{1 + \frac{1}{1}}$$

Partial Convergent

Definition (Partial Convergents)

For $0 \le n \le m$, the nth *convergent* of the continued fraction $[a_0, \ldots, a_m]$ is $[a_0, \ldots, a_n]$. These convergents for $n < m$ are also called *partial convergents*.



Partial Convergent

Proposition (Partial Convergents)

If p_n and q_n are defined by

$$p_0 = a_0, \; p_1 = a_1 a_0 + 1, \; p_n = a_n p_{n-1} + p_{n-2} \;\; \textit{for } n \leq 2$$

$$q_0 = 1, \; q_1 = a_1, \; q_n = a_n q_{n-1} + q_{n-2} \;\; \textit{for } n \leq 2$$

we have

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n}.$$

Partial Convergent

Proof.

We use induction. The assertion is obvious when $n = 0, 1$. Suppose the proposition is true for all continued fractions of length $n - 1$. □

Partial Convergent

Proof.

$$
\begin{aligned}
[a_0, \ldots, a_n] &= [a_0, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}] \\
&= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\
&= \frac{(a_{n-1} a_n + 1) p_{n-2} + a_n p_{n-3}}{(a_{n-1} a_n + 1) q_{n-2} + a_n q_{n-3}} \\
&= \frac{a_n(a_{n-1} p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1} q_{n-2} + q_{n-3}) + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}.
\end{aligned}
$$

Partial Convergent

Proposition

For $n \geq 0$ *with* $n \leq m$ *we have*

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$$

and

$$p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n.$$

Equivalently,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

and

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}}.$$

Partial Convergent

Proof.

The case for $n = 0$ is obvious from the definitions. Now suppose $n > 0$ and the statement is true for $n - 1$. Then

$$\begin{aligned} p_n q_{n-1} - q_n p_{n-1} &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - (a_n q_{n-1} + q_{n-2}) p_{n-1} \\ &= p_{n-2} q_{n-1} - q_{n-2} p_{n-1} \\ &= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\ &= -(-1)^{n-2} = (-1)^{n-1}. \end{aligned}$$

This completes the proof of Simple Continued Fraction. □

Partial Convergent

Theorem

If $[a_0, a_1, \ldots, a_m]$ *is a simple continued fraction, so each* a_i *is an integer, then the* p_n *and* q_n *are integers and the fraction* p_n/q_n *is in lowest terms.*

Partial Convergent

Theorem

If $[a_0, a_1, \ldots, a_m]$ *is a simple continued fraction, so each* a_i *is an integer, then the* p_n *and* q_n *are integers and the fraction* p_n/q_n *is in lowest terms.*

Proof.

It is clear that the p_n and q_n are integers, from the formula that defines them. If d is a positive divisor of both p_n and q_n, then $d \mid (-1)^{n-1}$, so $d = 1$. □

The Sequence of Partial Convergents

Let $[a_0, \cdots, a_n]$ be a continued fraction and for $n \leq m$ we write

$$c_n = [a_0, \cdots, a_n] = \frac{p_n}{q_n}$$

as the nth convergent.

The Sequence of Partial Convergents

Let $[a_0, \cdots, a_n]$ be a continued fraction and for $n \leq m$ we write

$$c_n = [a_0, \cdots, a_n] = \frac{p_n}{q_n}$$

as the nth convergent.

Proposition (How Convergents Converge)

The even indexed convergents c_{2n} increase strictly with n, and the odd indexed convergents c_{2n+1} decrease strictly with n. Also, the odd indexed convergents c_{2n+1} are greater than all of the even indexed convergents c_{2m}.

The Sequence of Partial Convergents

Proof.

The a_n are positive for $n \geq 1$, so the q_n are positive. By proposition, for $n \geq 2$,

$$c_n - c_{n-2} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}},$$

which proves the first claim. □

The Sequence of Partial Convergents

Proof.

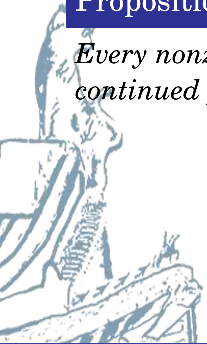
Suppose for the sake of contradiction that there exist integers r and m such that $c_{2m+1} < c_{2r}$. From the previous proposition implies implies that for $n \geq 1$,

$$c_n - c_{n-1} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

has sign $(-1)^{n-1}$, so for all $s \geq 0$ we have $c_{2s+1} > c_{2s}$. Thus it is impossible that $r = m$. If $r < m$, then by what we proved in the first paragraph, $c_{2m+1} < c_{2r} < c_{2m}$, a contradiction (with $s = m$). If $r > m$, then $c_{2r+1} < c_{2m+1} < c_{2r}$, which is also a contradiction (with $s = r$).

1. Finite Continued Fractions
   - Partial Convergent
   - The Sequence of Partial Convergents
   - Every Rational Number is Represented
2. Infinite Continued Fractions
   - The Continued Fraction Procedur
   - Convergence of Infinite Continued Fract

Every Rational Number is Represented



Proposition (Rational Continued Fractions)

Every nonzero rational number can be represented by a simple continued fraction.

Every Rational Number is Represented

Proof.

Without loss of generality, we may assume that the rational number is a/b, with $b \geq 1$ and $\gcd(a, b) = 1$.

$$
\begin{aligned}
a &= b \cdot a_0 + r_1, & 0 < r_1 < b \\
b &= r_1 \cdot a_1 + r_2, & 0 < r_2 < r_1 \\
&\dots & \\
r_{n-2} &= r_{n-1} \cdot a_{n-1} + r_n, & 0 < r_n < r_{n-1} \\
r_{n-1} &= r_n \cdot a_n + 0. &
\end{aligned}
$$

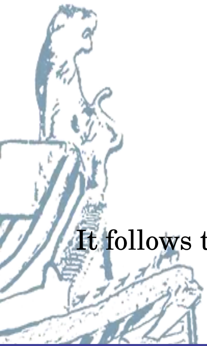
$\square$

Every Rational Number is Represented

Proof.

Note that $a_i > 0$ for $i > 0$ (also $r_n = 1$, since $\gcd(a, b) = 1$).
Rewrite the equations as follows:

$$
\begin{aligned}
a/b &= a_0 + r_1/b = a_0 + 1/(b/r_1),\\
b/r_1 &= a_1 + r_2/r_1 = a_1 + 1/(r_1/r_2),\\
r_1/r_2 &= a_2 + r_3/r_2 = a_2 + 1/(r_2/r_3),\\
&\ldots\\
r_{n-1}/r_n &= a_n.
\end{aligned}
$$



It follows that

$$\frac{a}{b} = [a_0, a_1, \ldots, a_n].$$

Continued Fractions

The Continued Fraction Procedur

Let $x \in \mathbf{R}$ and write

$$x = a_0 + t_0$$

with $a_0 \in \mathbf{Z}$ and $0 \leq t_0 < 1$. We call the number a_0 the *floor* of x, and we also sometimes write $a_0 = \lfloor x \rfloor$. If $t_0 \neq 0$, write

$$\frac{1}{t_0} = a_1 + t_1$$

The Continued Fraction Procedur

with $a_1 \in \mathbf{N}$ and $0 \leq t_1 < 1$. Thus $t_0 = \frac{1}{a_1+t_1} = [0, a_1 + t_1]$, which is a continued fraction expansion of t_0, which need not be simple. Continue in this manner so long as $t_n \neq 0$ writing

$$\frac{1}{t_n} = a_{n+1} + t_{n+1}$$

with $a_{n+1} \in \mathbf{N}$ and $0 \leq t_{n+1} < 1$. We call this procedure, which associates to a real number x the sequence of integers $a_0, a_1, a_2, \ldots$, the continued fraction process.

The Continued Fraction Procedur

Example

Let $x = \frac{8}{3}$. Then $x = 2 + \frac{2}{3}$, so $a_0 = 2$ and $t_0 = \frac{2}{3}$. Then $\frac{1}{t_0} = \frac{3}{2} = 1 + \frac{1}{2}$, so $a_1 = 1$ and $t_1 = \frac{1}{2}$. Then $\frac{1}{t_1} = 2$, so $a_2 = 2$, $t_2 = 0$, and the sequence terminates. Notice that

$$\frac{8}{3} = [2, 1, 2],$$

so the continued fraction procedure produces the continued fraction of $\frac{8}{3}$.

The Continued Fraction Procedur

Example

Let $x = \frac{1+\sqrt{5}}{2}$. Then

$$x = 1 + \frac{-1+\sqrt{5}}{2},$$

so $a_0 = 1$ and $t_0 = \frac{-1+\sqrt{5}}{2}$. We have

$$\frac{1}{t_0} = \frac{2}{-1+\sqrt{5}} = \frac{-2-2\sqrt{5}}{-4} = \frac{1+\sqrt{5}}{2},$$

The Continued Fraction Procedur

so $a_1 = 1$ and $t_1 = \frac{-1+\sqrt{5}}{2}$. Likewise, $a_n = 1$ for all n. As we will see below, the following exciting equality makes sense.

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$

The Continued Fraction Procedur

Example

Suppose $x = e = 2.71828182\ldots$. Using the continued fraction procedure, we find that

$$a_0, a_1, a_2, \ldots = 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots$$

The Continued Fraction Procedur

For example, $a_0 = 2$ is the floor of 2. Subtracting 2 and inverting, we obtain $1/0.718\ldots = 1.3922\ldots$, so $a_1 = 1$. Subtracting 1 and inverting yields $1/0.3922\ldots = 2.5496\ldots$, so $a_2 = 2$.
The 5th partial convergent of the continued fraction of e is

$$[a_0, a_1, a_2, a_3, a_4, a_5] = \frac{87}{32} = 2.71875,$$

which is a good rational approximation to e, in the sense that

$$\left|\frac{87}{32} - e\right| = 0.000468\ldots.$$

Note that $0.000468\ldots < 1/32^2 = 0.000976\ldots$, which illustrates the bound in Theorem.

The Continued Fraction Procedur

Let's do the same thing with $\pi = 3.14159265358979\ldots$.
Applying the continued fraction procedure, we find that the continued fraction of π is

$$a_0, a_1, a_2, \ldots = 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots$$

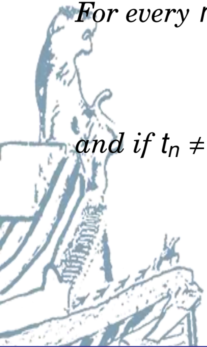
The first few partial convergents are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \ldots$$

These are good rational approximations to π; for example,

$$\frac{103993}{33102} = 3.14159265301\ldots.$$

Convergence of Infinite Continued Fract



Lemma

For every n *such that* a_n *is defined, we have*

$$x = [a_0, a_1, \ldots, a_n + t_n],$$

and if $t_n \neq 0$, *then* $x = [a_0, a_1, \ldots, a_n, \frac{1}{t_n}]$.

Convergence of Infinite Continued Fract

Proof.

We use induction. The statements are both true when $n = 0$. If the second statement is true for $n - 1$, then

$$\begin{aligned} x &= \left[a_0, a_1, \ldots, a_{n-1}, \frac{1}{t_{n-1}}\right] \\ &= [a_0, a_1, \ldots, a_{n-1}, a_n + t_n] \\ &= \left[a_0, a_1, \ldots, a_{n-1}, a_n, \frac{1}{t_n}\right]. \end{aligned}$$

Similarly, the first statement is true for n if it is true for $n - 1$. □

Convergence of Infinite Continued Fract

Theorem (Continued Fraction Limit)

Let $a_0, a_1, \ldots$ *be a sequence of integers such that* $a_n > 0$ *for all* $n \geq 1$*, and for each* $n \geq 0$*, set* $c_n = [a_0, a_1, \ldots a_n]$*. Then* $\lim_{n \to \infty} c_n$ *exists.*

Convergence of Infinite Continued Fract

Proof.

For any $m \geq n$, the number c_n is a partial convergent of $[a_0, \ldots, a_m]$. The even convergents c_{2n} form a strictly *increasing* sequence and the odd convergents c_{2n+1} form a strictly *decreasing* sequence. Moreover, the even convergents are all $\leq c_1$ and the odd convergents are all $\geq c_0$. Hence $\alpha_0 = \lim_{n\to\infty} c_{2n}$ and $\alpha_1 = \lim_{n\to\infty} c_{2n+1}$ both exist, and $\alpha_0 \leq \alpha_1$.

$$|c_{2n} - c_{2n-1}| = \frac{1}{q_{2n} \cdot q_{2n-1}} \leq \frac{1}{2n(2n-1)} \to 0,$$

so $\alpha_0 = \alpha_1$. $\square$

Convergence of Infinite Continued Fract

We define

$$[a_0, a_1, \ldots] = \lim_{n\to\infty} c_n.$$

Example

We illustrate the theorem with $x = \pi$.
Let c_n be the nth partial convergent to π. The c_n with n odd converge down to π

$$c_1 = 3.1428571\ldots,\ c_3 = 3.1415929\ldots,\ c_5 = 3.1415926\ldots$$

whereas the c_n with n even converge up to π

$$c_2 = 3.1415094\ldots,\ c_4 = 3.1415926\ldots,\ c_6 = 3.1415926\ldots.$$

Convergence of Infinite Continued Fract

Theorem

Let $a_0, a_1, a_2, \ldots$ *be a sequence of real numbers such that* $a_n > 0$ *for all* $n \geq 1$*, and for each* $n \geq 0$*, set* $c_n = [a_0, a_1, \ldots a_n]$*. Then* $\lim_{n \to \infty} c_n$ *exists if and only if the sum* $\sum_{n=0}^{\infty} a_n$ *diverges.*

Convergence of Infinite Continued Fract

Example

Let $a_n = \frac{1}{n\log(n)}$ for $n \geq 2$ and $a_0 = a_1 = 0$. By the integral test, $\sum a_n$ diverges, the continued fraction $[a_0, a_1, a_2, \ldots]$ converges. This convergence is very slow, since, e.g.

$$[a_0, a_1, \ldots, a_{9999}] = 0.5750039671012225425930\ldots$$

yet

$$[a_0, a_1, \ldots, a_{10000}] = 0.7169153932917378550424\ldots.$$

Convergence of Infinite Continued Fract

Theorem

Let $x \in \mathbf{R}$ *be a real number. Then* x *is the value of the (possibly infinite) simple continued fraction* $[a_0, a_1, a_2, \ldots]$ *produced by the continued fraction procedure.*

Convergence of Infinite Continued Fract

Theorem (Convergence of continued fraction)

Let $a_0, a_1, \ldots$ *define a simple continued fraction, and let* $x = [a_0, a_1, \ldots] \in \mathbf{R}$ *be its value. Then for all* m,

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{q_m \cdot q_{m+1}}.$$