Zeroes of 5-adic and 7-adic L-functions over cubic fields

A dissertation

submitted in partial fulfilment

of the requirements for the Degree

of

Bachelor of Science with Honours

at the

University of Waikato

by

Chris King (Qin Chao)

University of Waikato

2016

Abstract

First, we calculate the 5-adic and 7-adic λ-invariants for all cyclic cubic field K up to discriminant $< 10^7$. At the same time, we also determine the class number of $K(e^{2\pi i/p})$ for $p = 5$ and $p = 7$. If the λ-invariant is > 0, we then locate the zeroes of each p-adic L-function.

Acknowledgements

Firstly, I would like to express my gratitude to Daniel Delbourgo, who lead me to the field of algebraic number theory. Thank you, Daniel, for your patience and encouragement. You did not only teach me math, but also the attitude to work.

To the senior students of the Mathematics Department. Thank them for their help, as well as their beers.

To my parents for their love and support. Lastly, I wish to thank my lovely wife, Sarah. Thank you for taking care of the kids and letting me chase my dream.

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Chapter 1

p-adic Numbers

The p-adic numbers first appeared in Hensel's book *Theorie der algebraïschen Zavlen, Leipzig, 1908* [1]. Based on his mentor Kummer's work, Hensel developed the theory of p-adic numbers. Nowadays, the theory of p-adic numbers is playing an important role not only in number theory, but also in other branches of mathematics, even in physics. Furthermore, the p-adic idea can be applied in computer science, numerical analysis and simulations, uniform distribution of sequences, cryptography, combinatorics, automata theory and formal languages.

1.1 Valuations

In this section, we will discuss the connection between algebraic number theory and analysis. The concept of a valuation on a field becomes one of the most important tasks, since it connects these two branches. In particular, it will allow us to define the field of p-adic numbers, and the ring of p-adic integers.

Definition 1.1 *A valuation is a function from a field K, to the real number field $\mathbb{R}$, which satisfies the following axioms:*

1. *For all $a \in K$, $|a| \geq 0$, with $|a| = 0$ iff $a = 0$.*

2. *For all $a, b \in K$, $|ab| = |a||b|$.*

3. For all $a, b \in K$, $|a+b| \leq |a| + |b|$. (The triangle inequality)

Example 1.1 *(a) If $K = \mathbb{R}$, we define the normal absolute value as $|\pm a| = a$, for all $a \geq 0$.*

(b) If $K = \mathbb{C}$, then we take the normal modulus of complex numbers as the valuation of $\mathbb{C}$. e.g. $|a+bi| = \sqrt{a^2+b^2}$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

These two examples clearly satisfy all three axioms above, so they are both valuations.

We have some obvious consequences, namely (1) $|1| = 1$. (2) If $|a^n| = 1$, then $|a| = 1$. (3) $|-1| = 1$, and $|-a| = |a|$.

Definition 1.2 *Let p be a prime number and $a = p^k \frac{\alpha}{\beta} \in \mathbb{Q}$ such that $p \not| \alpha\beta$. We define the padic order to be*

$$ord_p a = k.$$

Furthermore, we define the p-adic absolute value as

$$|a|_p = \begin{cases} \frac{1}{p^{ord_p a}} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

It is obvious that the p-adic absolute value is a valuation. In fact, we also get the further property that $|a+b|_p \leq max\{|a|_p, |b|_p\}$.

Definition 1.3 *(Non-Archimedean Valuation) Let $|\cdot|$ be a valuation of a field K. If for any $a, b \in K$,*

$$|a+b| \leq max\{|a|, |b|\},$$

then we call the valuation $|\cdot|$ a non-Archimedean valuation.

It is not hard to show that a non-archimedean valuation satisfies the triangle inequality, as follows. Without loss of generality, assume $|a| \geq |b|$; then $max\{|a|, |b|\} = |a|$, and $|a+b| \leq max\{|a|, |b|\} = |a| \leq |a| + |b|$.

Lemma 1.2 *$| \ |_p$ is non-Archimedean.*

Proof. As we mentioned before, we can easily see that $|a|_p = 0$ iff $a = 0$.

If $a = 0$ or $b = 0$, $|a \cdot b|_p = 0 = |a|_p|b|_p$. If $a, b \neq 0$, then $|a \cdot b|_p = \frac{1}{p^{ord_p ab}} = \frac{1}{p^{ord_p a}} \cdot \frac{1}{p^{ord_p b}} = |a|_p|b|_p$.

If $a = 0$ or $b = 0$, or if $a + b = 0$, then the result holds trivially. So we assume that $a, b, a + b \neq 0$. Now we set $a = x_1/y_1$, and $b = x_2/y_2$. Then $a + b = (x_1y_2 + y_1x_2)/y_1y_2$, and $ord_p(a + b) = ord_p(x_1y_2 + y_1x_2) - ord_p y_1y_2$. That means

$$\begin{aligned} ord_p(a+b) &\geq min(ord_p x_1y_2, ord_p y_1x_2) - ord_p y_1 - ord_p y_2 \\ &= min(ord_p x_1 + ord_p y_2, ord_p y_1 + ord_p x_2) - ord_p y_1 - ord_p y_2 \\ &= min(ord_p x_1 - ord_p y_1, ord_p x_2 - ord_p y_2) \\ &= min(ord_p a, ord_p b). \end{aligned}$$

Then we deduce $|a + b|_p = \frac{1}{ord_p a+b} \leq max(\frac{1}{ord_p a}, \frac{1}{ord_p b}) = max(|a|_p, |b|_p)$, where $|a + b|_p \leq |a|_p + |b|_p$. □

Note that, if $| \cdot |$ is a non-archimedean valuation of a field K, then for any $a, b \in K$ with $|a| \neq |b|$, we have $|a + b| = max\{|a|, |b|\}$.

Theorem 1.3 *(Ostrowski 1918) Every nontrivial norm $|| \ ||$ on $\mathbb{Q}$ is either equivalent to $| \ |_p$ for some prime p, or is equal to $| \ |_\infty$.*

Proposition 1.4 *(Product Formula) Let $a \in \mathbb{Q}$ with $a \neq 0$. Then*

$$\prod_p |a|_p = 1$$

where $p \in \{primes\} \bigcup \{\infty\}$, and $|a|_\infty$ means the standard absolute value.

Proof. Assume that a is a positive integer, and

$$a = p_1^{k_1} p_2^{k_2} \cdots p_k^{k_k}$$

where p_i are primes. Then we have

$$\begin{cases} |a|_p = 1, \text{ if } p \neq p_i, \\ |a|_p = p^{-k_i}, \text{ if } p = p_i \\ |a|_\infty = p_1^{k_1} p_2^{k_2} \cdots p_k^{k_k}. \end{cases}$$

Since it is true for all the positive integers, it is also true for the negative ones, as $|a|_p = |-a|_p$ for all $p \in \{prmes\} \cup \{\infty\}$. One then extends it to $\mathbb{Q}$, using the multiplicativity of the valuation.

□

1.2 Field Completions

In this section, we will state some well known facts about field completions.

Definition 1.4 *Let K be a field with valuation $|\ |$. Then a sequence $\{a_n\} = \{a_1, a_2, \cdots\}$ is called Cauchy if for every positive number ϵ, there is a positive integer N such that for all natural numbers $m, n > N$, $|a_m - a_n| < \epsilon$.*

Definition 1.5 *Let K be a field with a valuation $|\ |$ on it. Then a sequence $\{a_n\} = \{a_1, a_2, \cdots\}$ is said to tend to a limit m, if for every $\epsilon > 0$, then there exists an $n_0(\epsilon)$ such that $|a_n - m| < \epsilon$ for all $n > n_0(\epsilon)$. If a sequence has a limit, then we call it convergent.*

Clearly, all convergent sequence in K are automatically Cauchy.

Definition 1.6 *In the field K, if every Cauchy sequence has a limit then we say the field K is complete with respect to the valuation $|\ |$.*

Using the same idea utilised in the construction of $\mathbb{R}$ from $\mathbb{Q}$, we can build up a unique extension field of K, say $\tilde{K}$, where $\tilde{K}$ is complete and every element in $\tilde{K}$ is a limit of Cauchy sequences from K.

Theorem 1.5 *Let K be a field with absolute value $|\,|$. There exists a unique extension field $\tilde{K}$ of K, with the properties:*

- *$\tilde{K}$ is complete w.r.t. $|\,|$;*
- *K is dense in $\tilde{K}$.*

Moreover we call $\tilde{K}$ the completion of K.

We can therefore represent each $a \in \tilde{K}$ as a limit, $a = \lim_{n=\infty} a_n$ with $a_n \in K$.

Corollary 1.6 *If $|\,|$ is a non-archimedean absolute value on K, then the natural extension of $|\,|$ to $\tilde{K}$ is also non-archimedean.*

Proof. Let $a, b \in \tilde{K}$. Then there exists $\{a_n\}, \{b_n\} \in K$, where $\{a_n\} \to a, \{b_n\} \to b$. Since $|\,|$ is a non-Archimedean absolute value on K, we simply take the limit of $|a_n + b_n| \leq max(|a_n|, |b_n|)$ with respect to n, which implies $|a + b| \leq max(|a|, |b|)$. So the extension of $|\,|$ to $\tilde{K}$ is also non-Archimedean.

□

1.3 p-adic Numbers

In this section, we will introduce p-adic numbers.

To get into this topic, we had better start with some elementary congruential arithmetic. Especially, we should keep our eyes on the integral polynomial congruences

$$f(x) \equiv 0 \pmod{p^n}, \text{where } n \in \mathbb{N}$$

for a fixed prime p. We start with $n = 1$, then try to build upon solutions for higher powers of p.

After that, we need to show the translation between the solutions and the power series of p. This is actually the foundation of the theory of p-adic numbers.

Theorem 1.7 *Let $f(x) \in \mathbb{Z}[x]$ be a nonzero polynomial, fix p as a prime, and $a \in \mathbb{N}$ such that*

$$f(a) \equiv 0 \ (mod\ p) \ \ and \ \ f'(a) \not\equiv 0 \ (mod\ p),$$

where $f'(a)$ is the derivative of $f(a)$. Then exist a sequence of integers $(a_0, a_1, a_2, \cdots)$ satisfying

$$f(a_n) \equiv 0 \ (mod\ p^{n+1}) \ \ and \ \ a_{n+1} \equiv a_n \ \ (mod\ p^{n+1}).$$

Further, a_n is unique modulo p^{n+1}.

Proof. We will prove this by induction. Begin with $a_0 = a$, which satisfies

$$f(a_0) = f(a) \equiv 0 \ (mod\ p).$$

So it is true for $n = 0$.

Now, suppose we have a sequence $(a_0, a_1, \cdots, a_n)$ satisfying the result. Then one can express

$$a_{n+1} = a_n + zp^{n+1}, \text{ for some } z \in \mathbb{Z}.$$

With this, the question left is whether we can find a value for z to make $f(a_{n+1}) \equiv 0 \ \ (mod\ p^{n+2})$.

In order to solve this problem, we use the Taylor expansion of $f(x)$ at the point $x = a_0$, i.e.

$$f(x) = f(a_n) + f'(a_n)(x - a_n) + \frac{1}{2}f''(a_n)(x - a_n)^2 + \cdots + \frac{1}{k!}f^{(k)}(a_n)(x - a_n)^k$$

Without loss of generality, assume there exists a $g(x) \in \mathbb{Z}[x]$. Such that

$$f(x) = f(a_n) + f'(a_n)(x - a_n) + g(x)f''(a_n)(x - a_n)^2;$$

substituting $x = a_n + zp^{n+1}$, we obtain

$$f(x) = f(a_n + zp^{n+1}) = f(a_n) + f'(a_n)(zp^{n+1}) + g(a_n + zp^{n+1})z^2p^{2n+2},$$

where $f(a_n) + f'(a_n)zp^{n+1} \equiv 0 \ (mod\ p^{n+2})$.

Also, we know that $f(a_n) \equiv 0 \ (mod \ p^{n+1})$. Then we can say $f(a_n) = cp^{n+1}$ for some $c \in \mathbb{Z}$. It follows that

$$cp^{n+1} + f'(a_n)zp^{n+1} \equiv 0 \ (mod \ p^{n+2}),$$

and dividing by p^{n+1},

$$c + f'(a_n)z \equiv 0 \ (mod \ p).$$

Since $f(a) \not\equiv 0 \ (mod \ p)$, then $f(a_n) \not\equiv 0 (mod \ p)$. In particular, $f(a_n)$ has an inverse modulo p, which is why we have a unique solution c to the congruence

$$z = -f'(a_n)^{-1}c.$$

Finally, putting z back into a_{n+1} and $f(a_{n+1})$, one obtains

$$f(a_{n+1}) \equiv 0 \ (mod \ p^{n+2}) \ \ and \ \ a_{n+1} \equiv a_n \ (mod \ p^{n+1})$$

as required . $\square$

Example 1.8 *Let us use this result to solve the congruences*

$$x^2 + 1 \equiv 0 \ (mod \ 5^{n+1}).$$

We start with $n = 0$*; the two solutions are clearly* $x = 2$ *and* $x = 3$*. Choosing* $x = 2$*, then* $a_0 = 2$*. Now, we use the fact* $a_{n+1} = a_n + zp^{n+1}$*, so* $a_1 = 2 + 5z$*. By Theroem1.7,*

$$(2 + 5z)^2 + 1 \equiv 0 \ \ (mod \ 25)$$

$$5 + 20z \equiv 0 \ \ (mod \ 25)$$

$$1 + 4z \equiv 0 \ \ (mod \ 5)$$

$$z \equiv 1 \ \ (mod \ 5)$$

so $a_1 = 2 + 5 \times 1 = 7$*. Similarly,* $a_2 = 7 + 25z$*, in which case*

$$\begin{aligned}(7+25z)^2+1 &\equiv 0 \pmod{125}\\ 50+350z &\equiv 0 \pmod{125}\\ 2+14z &\equiv 0 \pmod{5}\\ z &\equiv 2 \pmod{5}\end{aligned}$$

hence $a_2 = 7 + 25 \times 2 = 57$*. We can continue this process indefinitely to obtain a power series in p,*

$$x = \sum_{i=0}^{\infty} a_i p^i$$

which is the p-adic solution to $f(x) \equiv 0$*.*

Theorem 1.9 *Fix a prime number p. The field of p-adic numbers is the Cauchy completion of* $\mathbb{Q}$ *w.r.t.* $|\ |_p$*, we call it* $\mathbb{Q}_p$*.*

Formally, a p-adic number has the form

$$a_{-m}p^{-m} + \cdots + a_0 + a_1 p + \cdots + a_n p^n + \cdots, \textit{where } m, n \in \mathbb{N}.$$

Definition 1.7 *A* p-adic integer α *is a formal power series*

$$\alpha = \sum_{i=0}^{\infty} a_i p^i = a_0 + a_1 p + a_2 p^2 + \cdots$$

where $a_i \in \{0, 1, \cdots, p-1\}$*.*

The set of *p-adic integers* is denoted by $\mathbb{Z}_p$. If we truncated at nth term of the power series, we get

$$\alpha(mod\ p^n) = a_0 + a_1 p + \cdots + a_{n-1}p^{n-1},$$

which defines an element in $\mathbb{Z}/p^n\mathbb{Z}$, And so one obtains mappings

$$\mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}.$$

And clearly, $\mathbb{Z}$ is a *subring* of $\mathbb{Z}_p$.

For example, a 7-adic solution of $3x = 5$ is

$$4 + 2 \times 7 + 2 \times 7^2 + 2 \times 7^3 + 2 \times 7^4 + \cdots$$

and a 5-adic solution of $x^2 = 11$ is given by

$$1 + 5 + 2 \times 5^2 + 2 \times 5^5 + 3 \times 5^7 + 3 \times 5^8 + 2 \times 5^9 + 5^{11} + \cdots$$

If we let $\alpha = \lim_{n=\infty} a_n$ in Theorem 1.7, then we obtain

Theorem 1.10 *(Hensel's Lemma) If $f(X) \in \mathbb{Z}_p[X]$ and $a \in \mathbb{Z}_p$ satisfies*

$$f(a) \equiv 0 \ (mod\, p) \ \ and \ \ f'(a) \not\equiv 0 \ (mod\, p)$$

then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $f(\alpha) = 0$ and $\alpha = a \ (mod\, p)$.

Example 1.11 *Let $f(X) = X^2 - 7$. Then $f(1) = -6 = 0 \ (mod\ 3)$ and $f'(1) = 2 \neq 0 \ (mod\ 3)$, so Hensel's lemma tells us there is a unique 3-adic integer α such that $\alpha^2 = 7$ and $\alpha = 1 \ (mod\, 3)$.*

Example 1.12 *Let us write -1 as a p-adic integer. Then*

$$\begin{aligned} -1 &= p - 1 - 1 \cdot p \\ &= p - 1 + (p-1) \cdot p - p^2 \\ &= p - 1 + (p-1) \cdot p = (p-1) \cdot p^2 - p^3 \\ &= \cdots \\ &= p - 1 + (p-1) \cdot p + (p-1) \cdot p^2 + (p-1) \cdot p^3 + \cdots \end{aligned}$$

Lemma 1.13 *If p is a prime number, then it is also a prime element in $\mathbb{Z}_p$.*

Proof. Suppose that $a, b \in \mathbb{Z}_p$ and $p|ab$. Let $a = a_0 + a_1 p + a_2 p^2 + \cdots$ and $b = b_0 + b_1 p + b_2 p^2 + \cdots$, where $0 \leq a_i, b_i < p$. Then

$$\begin{aligned} ab &= (a_0 + a_1 p + a_2 p^2 + \cdots)(b_0 + b_1 p + b_2 p^2 + \cdots) \\ &= a_0 b_0 + p(a_0 b_1 + a_1 b_0) + \cdots \end{aligned}$$

which is a multiple of p.

From the equation above, we know that $p|a_0b_0$, which implies that $p|a_0$ or $p|b_0$. But we already specified that $0 \leq a_i, b_i < p$, so the only possibilities are $a_0 = 0$ or $b_0 = 0$. In each case $a = p(a_1 + a_2p + \cdots)$ or $b = p(b_1 + b + 2p + \cdots)$ respectively, which means that $p|a$ or $p|b$. Hence p is a prime element in $\mathbb{Z}_p$. □

Lemma 1.14 *If $\alpha \in \mathbb{Z}_p$ and $p \not| \alpha$, then α is a **unit**.*

Proof. Assume that $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots$ with $0 \leq \alpha_i < p$. Since $p \not| \alpha$, that means $\alpha_0 \neq 0$. Assume there exists another p-adic integer $\beta = \beta_0 + \beta_1 p + \beta_2 p^2 + \cdots$ such that $\alpha\beta = 1$. Then

$$\alpha\beta = (\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots)(\beta_0 + \beta_1 p + \beta_2 p^2 + \cdots)$$

in which case

$$\begin{aligned} 1 = \alpha\beta &\equiv \alpha_0\beta_0 (mod\, p) \\ &\equiv \alpha_0\beta_0 + p(\alpha_1\beta_0 + \alpha_0\beta_1)(mod\, p^2). \end{aligned}$$

Since $1 \equiv \alpha_0\beta_0(mod\, p)$, we can write that $\alpha_0\beta_0 - 1 = p\gamma_0$. Then $0 \equiv p\gamma_0 + p(\alpha_1\beta_0 + \alpha_0\beta_1)(mod\, p^2)$ so $\alpha_1\beta_0 + \alpha_0\beta_1 + \gamma_0 \equiv 0(mod\, p)$, and then $\alpha_0\beta_1 \equiv -\gamma_0 - \alpha_1\beta_0(mod\, p)$ has a unique solution, as well as β_1.
??Using this method, we could find $\beta_0, \beta_1, \cdots, \beta_n$, then $\alpha\beta \equiv 1(mod\, p^n)$ for every n. That means $p^n|(\alpha\beta - 1)$ for every n, so the only possible way to satisfy that is $\alpha\beta = 1$. □

Lemma 1.15 Every nonzero p-adic integer has a unique form $\alpha = p^n a$, *where integer $n \geq 0$ and the unit $a \in \mathbb{Z}_p^\times$*

Proof. Assume that $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots$ with $0 \leq \alpha_i < p$. Let n be the smallest exponent for which a_n is nonzero. Then

$$\begin{aligned} \alpha &= \alpha_n p^n + \alpha_{n+1} p^{n+1} + \cdots \\ &= p^n(\alpha_n + \alpha_{n+1} p + \cdots) \end{aligned}$$

and since $p \not| \alpha_n$, the p-adic number $(\alpha_n + \alpha_{n+1}p + \cdots)$ is a unit. □

Lemma 1.16 *The ring $\mathbb{Z}_p$ is a integral domain.*

Proof. Assume that $a, b \in \mathbb{Z}_p - \{0\}$, and $ab = 0$. Now, suppose that $a \neq 0, b \neq 0$; then we may write $a = p^m\alpha$ and $b = p^n\beta$, where α, β are units. Since $\alpha\beta$ is also a unit, one obtains

$$\begin{aligned} ab &= p^m\alpha \cdot p^n\beta \\ &= p^{m+n}\alpha\beta \neq 0 \end{aligned}$$

which yields a contradiction. Thus $ab = 0$ implies $a = 0$ or $b = 0$. □

Theorem 1.17 *The p-adic series*

$$\sum_{n=1}^{\infty} a_n, \ \textit{where } a_n \in \mathbb{Q}_p$$

converges iff $|a_n|_p \to 0$ as $n \to \infty$.

Proof. It is clear to see that if the series $\sum_{n=1}^{\infty} a_n$ converges, that implies $|a_n|_p \to 0$ when $n \to \infty$. Conversely, let's put $S_k = \sum_{n=0}^{k} a_n$. This new sequence $\{S_k\}_{k=1}^{\infty}$ is Cauchy, by the strong triangle inequality. In fact, if $j > k$ then we have

$$\begin{aligned} |S_j - S_k|_p &= |a_{k+1} + a_{k+2} + \cdots + a_j|_p \\ &\leq max(|a_{k+1}|_p, \cdots, |a_j|_p). \end{aligned}$$

and as $|a_n|_p \to 0$, the S_k's converge. □

As an illustration, the sequence

$$\sum_{n=0}^{\infty} p^n$$

converges to $\frac{1}{1-p}$.

Theorem 1.18 *Let $a_{ij} \in \mathbb{Q}_p$, and also for any i, $lim_{j\to\infty} a_{ij} = 0$. Suppose that for every j, we have $lim_{i\to\infty} a_{ij} = 0$. Then the two series*

$$\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{ij}) \textit{ and } \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} a_{ij})$$

both converge, and they are equal to each other.

Proof. From the observation above, we realized that both the sums should be converged. That is because $i, j \leq max(M, M_1) = M_0$. So we have

$$\Big|\sum_{k=0}^{\infty} a_{ik}\Big|_p < \epsilon \text{ and } \Big|\sum_{l=0}^{\infty} a_{lj}\Big|_p < \epsilon.$$

Now we take the differences

$$\Big|\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{ij}) - \sum_{i=0}^{M_0}(\sum_{j=0}^{M_0} a_{ij})\Big|_p = \Big|\sum_{i=0}^{M_0}(\sum_{j=M_0+1}^{\infty} a_{ij}) - \sum_{i=M_0+1}^{\infty}(\sum_{j=0}^{\infty} a_{ij})\Big|_p < \epsilon.$$

Similarly, we have

$$\Big|\sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} a_{ij}) - \sum_{j=0}^{M_0}(\sum_{i=0}^{M_0} a_{ij})\Big|_p < \epsilon.$$

Then we get the result that, for any $\epsilon > 0$, we have

$$\Big|\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{ij}) - \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} a_{ij})\Big|_p \leq max(\Big|\sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{ij}) - S\Big|_p, \Big|S - \sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} a_{ij})\Big|_p) \leq \epsilon$$

where $S = \sum_{i=0}^{M_0} \sum_{j=0}^{M_0} a_{ij}$. □

Chapter 2

Dirichlet L-series

The Dirichlet L-series are named after Peter Gustav Lejeune Dirichlet, who introduced them to in 1837. One of the simplest questions is whether there are infinitely many primes congruent to $a(mod\ m)$. While the answer is "Yes", i.e. there are infinitely many primes if and only if a and m are coprime to each other, history tells us it is much harder than proving there are infinitely many primes. In this chapter, we will introduce Dirichlet characters, L-series and Bernoulli numbers first, then use them to construct the p-adic L-function.

2.1 Dirichlet Characters

A Dirichlet character is a multiplicative homomorphism χ: $(\mathbb{Z}/n\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$, with the following description.

Definition 2.1 *For a positive integer n, a Dirichlet character mod n is a function χ: $\mathbb{Z} \to \mathbb{C}$ which has following properties*

- *n-periodic: if $a \equiv a'$ mod n then $\chi(a) = \chi(a')$;*
- *supported on the integers coprime to n: $(a, n) = 1$ if and only if $\chi(a) \neq 0$;*
- *multiplicative: $\chi(ab) = \chi(a)\chi(b)$.*

the smallest such period n the **conductor** of the character, denote f_χ. For convenience, we say χ is even if $\chi(-1) = 1$, While if $\chi(-1) = -1$ then we call

χ odd.

Definition 2.2 *A Dirichlet character χ (mod n) is called primitive if it is not induced by any character of a modulus m with $m < n$.*

Let n be some positive integer, if $z^n = 1$, then we call complex number z a root of unity. For instance, numbers of form $e^{2\pi ia/n}$ are primitive n-th roots of unity.

Example 2.1 *We define*

$$\chi_0(a) = \begin{cases} 1 & \text{if } (a,n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and χ_0 here is called the trivial or principal character (with modular n).

Of course, we care more about non-trivial or non-principal characters!

Example 2.2 *(Legendre symbols as characters) Let p be a odd prime, and $(\frac{n}{p})$ denote the Legendre symbol modulo p. We define that $\chi(n) = (\frac{n}{p})$, so that*

$$\chi(n) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue mod } p \\ 0 & \text{if } gcd(n,p) > 1 \\ -1 & \text{otherwise} \end{cases}$$

Then $\chi(n)$ is a character modulo p. (It is not hard to prove, all three properties follow immediately from the definition of the Legendre symbol.)

Theorem 2.3 *The number of Dirichlet characters with conductor n is $\phi(n)$.*

Proof. Since the group $\mathbb{Z}/p\mathbb{Z}$ is cyclic, we write $\mathbb{Z}/p\mathbb{Z} = \{g, g^2, \cdots, g^{p-1}\}$ for some g. Let $x \in \mathbb{Z}/p\mathbb{Z}$, then $x \equiv g^k$ for some k depending on x. As $\chi(g^k) = \chi(g)^k$, we can determine the Dirichlet characters of all the elements one we determine the value for generator $g(\chi(0) = \chi(n) = 0)$. Since $gcd(g,n) = 1$, then $\chi(g)$ is a root of unity. As $g^{p-1} \equiv 1(mod\ p)$ and $\chi(1) = 1$, $\chi(g)^{p-1} = 1$. Therefore $\chi(g) = e^{2\pi ai}/(p-1)$, where $a \in \{1, 2, \cdots, p-1\}$. It completes the proof by noting each of the possible choice of a. $\square$

Theorem 2.4 *(Orthogonality relations) Let n be a positive integer, then the Dirichlet character χ modulo n has the property*

$$\sum_{a \bmod n} \chi(a) = \begin{cases} \phi(n) & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 L-series

In modern number theory, L-functions can be associated to several categories of mathematical objects: number fields, Galois representations, elliptic curves, modular forms. An L-series is a power series, normally convergent on a half-plane, that gives rise to an L-function via analytic continuation.

Definition 2.3 *The Riemann zeta function is defined as*

$$\zeta(s) = \sum \frac{1}{n^s}$$

for $s \in \mathbb{C}$ with $Re(s) > 1$.

Lemma 2.5 *(Euler) For $Re(s) > 1$,*

$$\zeta(s) = \sum \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

If we take a deeper look at the R.H.S. When $s = 1$, Euler's formula become $\prod_p \frac{p}{p-1}$. And it diverges, which implies there are infinitely many primes. Another important feature of $\zeta(s)$ is its *functional equation*, which is the bridge between the values at s and the values at $1 - s$.

Definition 2.4 *$\zeta(s)$ has the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \Gamma(1 - s) \zeta(1 - s),$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-s}\, dt$ with the property $\Gamma(s + 1) = s\Gamma(s)$.

We can rewrite the functional equation in terms of the completed zeta function

$$\xi(s) = \frac{s(s-1)}{2\pi^{s/2}} \Gamma(\frac{s}{2}) \zeta(s),$$

in which case it becomes $\xi(s) = \xi(1 - s)$.

Example 2.6 *Here some examples of the specific values of zeta function:*

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

and

$$\zeta(12) = \frac{691\pi^2}{638512875}$$

Example 2.7 *Another "crazy" example is $\zeta(-1)$. Via analytic continuation, one can show that*

$$\zeta(-1) = -\frac{1}{12}$$

gives a finite answer to the divergent series $1+2+3+4+\cdots$. The result has been already used in string theory.

The zeroes of Riemann zeta function are intimately connected with prime numbers. Recall that the Chebyshev function is defined as

$$\psi(x) = \sum_{p^k \leq x} log\, p,$$

which counts the number of prime powers up to x. It also has an explicit form

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - log\, 2\pi,$$

where ρ runs over the zeroes of $\zeta(x)$. In other words, once we locate the zeros, then we would know exactly where the prime powers are. The formula has already been used to deduce the prime number theorem:

$$\pi(x) := \#\{p : p \leq x\} \sim \frac{\psi(x)}{log\, x} \sim \frac{x}{log\, x}.$$

Hadamard and de la Vallée-Poussin came up with the proof first in 1896, then in 1940s Selberg and Erdős found out a different proof. The unsolved Riemann Hypothesis assets that all nontrivial zeroes actually lie on the critical line $Re(s) = 1/2$. If someone could prove it, then it would show us a perfect bound on the error term in prime number theorem.

Definition 2.5 *Dirichlet series is defined as*

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

One can see that if $a_n = 1$, $D(s) = \zeta(s)$ is the Riemann zeta function.

Definition 2.6 *We define the L-function assoerated to the Dirichlet character χ to be*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $Re(s) \geq 1$.

As χ is multiplicative, we also have the Euler product expansion

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

Dirichlet L-functions allow us to study primes lying in arithmetic progressions, since Dirichlet characters mod n enable us to work in different residue classes mod n. Dirichlet showed that for every nontrivial χ, we always have $L(1, \chi) \neq 0$. With this result, he also proved that there were infinity many primes of the form $\alpha n + \beta$, where α, β are integers, and β is coprime to n.

Not surprisingly, Dirichlet L-functions also have functional equations. If χ is a primitive Dirichlet character mod n, then the completed Dirichlet L-function is defined as

$$\Lambda(s, \chi) = \left(\frac{n}{\pi}\right)^{\frac{s+\epsilon}{2}} \Gamma\left(\frac{s+\epsilon}{2}\right) L(s, \chi),$$

where $\epsilon \in \{0, 1\}$ is the parity of $\chi(-1)$. i.e. if $\chi(-1) = 1$, then $\epsilon = 0$, whilst if $\chi(-1) = -1$, then $\epsilon = 1$. Reorganizing, we obtain

$$\Lambda(s, \chi) = (-i)^{\epsilon} \sqrt{n} \Big(\sum_{m=1}^{m} \chi(n) e^{2\pi i m/n} \Big) \Lambda(1 - s, \overline{\chi}),$$

where $\overline{\chi}$ is the complex conjugate of χ. Note that $\Lambda(s, \chi)$ is related to $\Lambda(1 - s, \overline{\chi})$, not $\Lambda(1 - s, \chi)$. (However if χ is real valued or χ^2 is trivial, then $\chi = \overline{\chi}$.)

2.3 Bernoulli Numbers

The Bernoulli numbers made their first appearance in *Ars Conjectandi* [2] in 1713, which is 8 years after Jacob Bernoulli died. Bernoulli was trying to find out a general formula for the sum

$$s_k(n) = 1^k + 2^k + \cdots + (n-1)^k.$$

However, very similar work also solved by the Indian mathematian Aryabhata in the 5th century A.D, though he only provided the formulae for $k = 1, 2$ and 3. Bernoulli's idea was calculating the first few s_k, then use the recurrence formula to calculate the next.

Definition 2.7 *The Bernoulli numbers B_n are defined by the expansion*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Lemma 2.8 *We have $B_0 = 1$, and for $n \geq 1$ the recurrence relation*

$$B_n = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k.$$

For example, the first few Bernoulli Numbers are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = -\frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$$

Theorem 2.9 *$B_n = 0$, if n is an odd number > 1.*

Proof. Since

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{-t}{e^{-t} - 1} + \frac{-t}{2}$$

it follows that

$$\begin{aligned}
-t &= \frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1} \\
-t &= \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n \frac{(-t)^n}{n!} \\
-t &= -t + 2\frac{B_3}{3!} t^3 + 2\frac{B_5}{5!} t^5 + 2\frac{B_7}{7!} t^7 + \cdots \\
0 &= \frac{B_3}{3!} t^3 + \frac{B_5}{5!} t^5 + \frac{B_7}{7!} t^7 + \cdots
\end{aligned}$$

Since the series expansion is equal to 0, all the coefficients on the R.H.S. must be zero.

□

Definition 2.8 *We define the n-th Bernoulli polynomial as*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

As $B_0 = 1$, then we can see $B_n(x)$ is a monic polynomial of degree n. Moreover

$$B_0(x) = 1, B_1(x) = x + \frac{1}{2}, B_2(x) = x^2 + x + \frac{1}{6}, \cdots$$

We can now describe the general solution for $s_k(n)$ as

$$s_k(n) = \frac{1}{k+1}\big(B_{k+1}(n) - B_{k+1}\big).$$

Definition 2.9 *The n-th Bernoulli polynomial has the generating function*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(xt^n)}{n!}.$$

Lemma 2.10 *For all $x \in \mathbb{R}$,*

$$B_n(1-x) = (-1)^n B_n(x).$$

Proof. From the above definition,

$$\begin{aligned} \frac{te^{(1-x)t}}{e^t - 1} &= \frac{te^{t-xt}}{e^t - 1} \\ &= \frac{te^{-xt}}{1 - e^{-t}} \\ &= \frac{(-t)e^{-xt}}{e^{-t} - 1} \\ &= \sum_{n=0}^{\infty} B_n(x) \frac{(-t)^n}{n!}. \end{aligned}$$

As a consequence, $B_n(1-x) = (-1)^n B_n(x)$. □

In 1734, Euler discovered the amazing formula,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

and more generally,

$$\sum_{n=-\infty}^{\infty}{}^{\heartsuit} \frac{1}{n^k} = -\frac{(2\pi\sqrt{-1})^k B_k}{k!}, \; k \geq 2,$$

where $\heartsuit$ means we omit $n = 0$. While k is odd, we get zero for both sides, so we can not get any information for the values of $\zeta(s)$. On the other hand, when k is even, we can get a wonderful consequence: $|B_{2k}/2k| \to \infty$ as $k \to \infty$.

Now, exploiting the functional equation of Riemann zeta function, we can rewrite Euler's formula in the simpler form

$$\zeta(1-k) = -\frac{B_k}{k}.$$

Definition 2.10 *We define the generalized Bernoulli numbers $B_{n,\chi}$ by the formula*

$$B_{n,\chi} = f^{n-1} \sum_{a (mod\ f)} \chi(a) B_n(\frac{a}{f}),$$

where χ is a Dirichlet character, f is the conductor of χ and B_n is the n-th Bernoulli number.

And for integer $n \geq 1$, we have

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}.$$

Another interesting property of Bernoulli numbers is

Theorem 2.11 *(von Staudt- Clausen) For each even k, we have*

$$B_k + \sum_{(p-1)|k} \frac{1}{p} \in \mathbb{Z}.$$

The proof can be found in [3].

2.4 p-Adic L-Functions

In this section, we will introduce the p-adic L-function, and it is based on Washington's book [4]. We can not construct the p-adic L-function directly from the classical L-function, since the usual L-series does not converge p-adically. But, by adding a fudge factor, the p-adic L-function interpolates the classical L-function on the negative integers. Kubota and Leopoldt first introduced this function in 1964, and later Iwasawa constructed it in a different way by using the relationship with cyclotomic fields. We start with some result form p-adic interpolation.

Definition 2.11 *Define the p-adic logarithm as the power series*

$$log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

The following two propositions are the keys to construct the p-adic L-function, the proofs can be found in [4].

Proposition 2.12 *There exists a unique extension of log_p to all of $\mathbb{C}_p \setminus \{0\}$ such that $log_p(p) = o$ and $log_p(xy) = log_p(x) + log_p(y)$ for all $x, y \in \mathbb{C}_p \setminus \{0\}$.*

Proposition 2.13 *Suppose that $r < p^{-1/(p-1)} < 1$ and for $x \in \mathbb{C}_p$,*

$$f(x) = \sum_{n=}^{\infty} \binom{x}{n} a_n$$

with $|a_n|_p \leq Mr^n$ for some M. Then, f(x) can be expressed as a power series with radius of convergence at least $R = (rp^{1/(p-1)})^{-1}$.

Definition 2.12 *We define the Hurwitz zeta function by*

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for real number is $0 < x \leq 1$.

From this definition, we can deduce that

$$\zeta(1-n,x) = -\frac{B_n(x)}{n}.$$

We can express a Dirichlet L-function as a linear combination of Hurwitz zeta functions. via

$$L(s,\chi) = \sum_{a(mod\ f)} \chi(a) \sum_{n=a(mod\ f)} \frac{1}{n^s}$$

and then focusing on the second summation. Take a look at $\sum_{n=a(mod\ f)} \frac{1}{n^s}$,

$$\sum_{n=a(mod\ f)} \frac{1}{n^s} = f^{-s} \sum_{j=0}^{\infty} \frac{1}{(j+\frac{a}{f})^s} = f^{-s}\zeta(s, \frac{a}{f}).$$

Putting it back to the equation above, we deduce that

$$L(s,\chi) = f^{-s} \sum_{a(mod\ f)} \chi(a)\zeta(s, \frac{a}{f}).$$

Furthermore,

$$L(1-n,\chi) = (-1)^n \chi(-1) L(1-n,\chi)$$

so, if χ and n have opposite parity, then $L(1-n,\chi) = 0$. Otherwise, we get the following formula from the functional equation

$$\Big|L(n,\chi)\Big| = \Big|\tau(\chi)\Big|(\frac{2\pi}{f})^n \Big|\frac{B_{n,\bar{\chi}}}{n!}\Big|, \text{ for } n \geq 1$$

where $\tau(\chi) = \sum_{a(mod\ f)} \chi(a) e^{2\pi i a/f}$ is the Gauss sum(which is why if $n \geq 1$, $L(1-n,\chi) \neq 0$ if and only if $\chi(-1) = (-1)^n$).

Definition 2.13 *Setting $x = \frac{a}{F}$, we can define Hurwitz zeta function by*

$$H(s,a,F) = \sum_{n=0}^{\infty} (n + \frac{a}{F})^{-s}, 0 < a \leq F$$

We shall replace this with a p-adic avatar.

Definition 2.14 *The Teichmüller character is the homomorphism of multiplicative groups*

$$\omega : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$$

such that $\omega(a) = a(mod\ p)$ is the unique $(p-1)st$ root of unity in $\mathbb{Z}_p$.

For $a \in \mathbb{Z}_p$, we define $\langle a \rangle = \omega^{-1}(a)a$.

Definition 2.15 *We define the p-adic Hurwitz zeta function by*

$$H_p(s,a,F) = \frac{1}{s-1}\frac{1}{F}\langle a \rangle^{1-s} \sum_{n=0}^{\infty} \binom{1-s}{n} (F/a)^n B_n, \text{ for all } s \in \mathbb{Z}_p.$$

Now, let us set

$$q = \begin{cases} p & \text{if } p \neq 2 \\ 4 & \text{if } p = 2 \end{cases}$$

Theorem 2.14 *Suppose $q|fF$ and $(a,p)=1$. Then $H_p(s,a,F)$ is analytic for all $s \neq 1$ and $s \in \mathbb{C}_p$ satisfying $|s|_p < qp^{-1/(p-1)}$. At $s=1$, it has a simple pole with residue $1/F$.*

Proof. We apply Proposition 2.13 to the series

$$\sum_{n=0}^{\infty} \binom{s}{n} (F/a)^n B_n.$$

Observe that by von Staudt-Clausen theorem, we have the following property for odd p,

$$\left|(F/a)^n B_n\right|_p \leq p^{-(k-1)} = p(1/p)^n$$

so that we can take $r = 1/p$ and $M = p$ in Proposition 2.13. A stronger estimate holds for $p = 2$. And in both case, the series above converges in $A = \{s \in \mathbb{C}_p : \quad |s|_p < qp^{-1/(p-1)}\}$. As $1 \in A$ and any point of a p-adic disc is its 'center', A is the same the set $\{s \in \mathbb{C}_p : \quad |1-s|_p < qp^{-1/(p-1)}\}$. This proves that

$$\sum_{n=0}^{\infty} \binom{1-s}{n} (F/a)^n B_n$$

is analytic in A. We know that $\langle a\rangle^s$ is analytic in A and hence $\langle a\rangle^{1-s}$. It is clear that at $s=1$, $H_p(s,a,F)$ has a simple pole at $s=1$ with residue $1/F$. This completes the proof.

□

Now, we can finally construct the p-adic L-function.

Theorem 2.15 *Let χ be a Dirichlet character of conductor f and let F be a multiple of q and f. Then there exists a p-adic meromorphic (analytic if $\chi \neq 1$) function $L_p(s,\chi)$ define on*

$$A = \{s \in \mathbb{C}_p : \quad |s|_p < qp^{-1/(p-1)}\}$$

such that

$$L_p(1-n,\chi) = -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n}, \quad n \geq 1.$$

If $\chi = 1$, then $L_p(s,1)$ is analytic in A except for a simple pole at $s = 1$ with residue $(1 - 1/p)$. Furthermore, we have the formula

$$L_p(s,\chi) = \frac{1}{F}\frac{1}{s-1}\sum_{a=1\&(a,p)=1}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{n=0}^{\infty}\binom{1-s}{n}(\frac{F}{a})^n B_n.$$

Proof. $L_p(s,\chi)$ is given by

$$L_p(s,\chi) = \sum_{a=1\&(ap)=1}^{F}\chi(a)H_p(s,a,F).$$

By the last theorem, $L_p(s,\chi)$ is analytic in A, except $s = 1$. When $s = 1$, $L_p(s,\chi)$ has residue

$$\frac{1}{F}\sum_{a=1\&(ap)=1}^{F}\chi(a).$$

Then, if $\chi = 1$, the residue is $\frac{p-1}{p}$. If $\chi \neq 1$, the residue becomes

$$\frac{1}{F}\sum_{a=1}^{F}\chi(a) - \frac{1}{F}\sum_{b=1}^{F/p}\chi(pb).$$

The first sum is zero. If $p|f$, then $\chi(pb) = 0$ for all b, hence the second sum is also zero. If $(p,f) = 1$, then $f|F/p$ so again the second sum is zero. Furthermore,

$$\begin{aligned}L_p(1-n,\chi) &= \sum_{a=1\&(ap)=1}^{F}\chi(a)H_p(1-n,a,F)\\ &= -\frac{1}{nF}\sum_{a=1\&(a,p)=1}^{F}\chi(a)\langle a\rangle^m\sum_{n=0}^{m}\binom{m}{n}(\frac{F}{a})^n B_n\end{aligned}$$

. Rewriting the inner sum

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}(\frac{F}{a})^n B_n &= \sum_{n=0}^{m}\binom{m}{n}(\frac{F}{a})^{m-n}B_{m-n}\\ &= (\frac{F}{a})^m\sum_{n=0}^{m}\binom{m}{n}(\frac{a}{F})^n B_{m-n}\end{aligned}$$

Putting the Teichmüller character in, we deduce

$$\begin{aligned}L_p(1-n,\chi) &= -\frac{F^{n-1}}{n}\sum_{a=1\&(a,p)=1}^{F}\chi\omega^{-n}(a)B_n(a/F)\\ &= -\frac{F^{n-1}}{n}\sum_{a=1}^{F}\chi\omega^{-n}(a)B_n(a/F) + \frac{F^{n-1}}{n}\sum_{b=1}^{F/p}\chi\omega^{-n}(pb)B_n(b/(F/p))\\ &= -\frac{B_{n,\chi\omega^{-n}}}{n} + \frac{\chi\omega^{-n}(p)p^{n-1}B_{n,\chi\omega^{-n}}}{n}\\ &= -(1-\chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n}.\end{aligned}$$

□

Chapter 3

Cubic Fields

The main aim of this essay is to compute the cyclotomic λ-invariant of a cubic field. We start by introducing some basic background.

3.1 What are Cubic Fields?

By definition, a cubic field is an algebraic number field of degree three over $\mathbb{Q}$. We start by considering polynomials of degree 3. Without loss of generality, assume the polynomial has the form

$$P(X) = X^3 + AX + B,$$

where A and B are rational numbers. Assuming $P(X)$ is irreducible over $\mathbb{Q}$, a zero ξ of $P(X)$ generates a cubic number field $L = \mathbb{Q}(\xi)$ after adjoining ξ to $\mathbb{Q}$.

Definition 3.1 *A field K is called a cubic field if it is a number field of degree $[K : \mathbb{Q}] = 3$. In particular, K is isomorphic to a field of the form $\mathbb{Q}[X]/P(X)$.*

Depending on the number of real zeroes of $P(X)$, we distinguish two different types of cubic fields. If $P(X)$ has three real roots, we call K a totally real cubic field. On the other hand, if $P(X)$ has a non-real root, then K is called a simple real (or complex) cubic field.

There are some special cubic fields; for example, K is a pure cubic field if it can be obtained as $\mathbb{Q}(n^{\frac{1}{3}})$, where $n \in \mathbb{N}$ and $n^{\frac{1}{3}}$ is real.

3.2 Cyclic Cubic Field

Another special cubic field is called cyclic cubic field.

Definition 3.2 *A field K is called a cyclic cubic field if it contains all three roots of $P(X)$. Furthermore, K is also normal and its Galois group is isomorphic to cyclic group $\mathbb{Z}/3\mathbb{Z}$.*

Recall that the discriminant of $P(X)$ is $\triangle = 4A^3 - 27B^2$.

Theorem 3.1 *If $K = \mathbb{Q}(\theta)$ is a cubic field, then K is a cyclic cubic field if and only if the discriminant of $P(X)$ is a square.*

Our next task is to give a general equation for such cyclic cubic fields. First, any cubic field has at least one real embedding. Then if K is a cyclic cubic field, that means all the roots of $P(X)$ must be real since they are all in K. So a cyclic cubic field is automatically totally real. Now, let $\xi_3 = e^{2\pi i/3}$ be a primitive cube root of unity. Clearly, $\theta \notin K$, since K is totally real; in particular, the extension field $K(\theta)$ must be a sextic field.

Theorem 3.2 *Let K be a cyclic cubic field with conductor f, where f is congruent to 1 modulo 3. Then there exist a unique integer $a \equiv 2 \pmod 3$ for each f such that $K = \mathbb{Q}(\theta)$ where θ is a root of*

$$P(X) = X^3 - 3fX - fa.$$

More generally, Hasse [5] indicated that one can write $f = (a^2 + 3b^2)/4$, where a and b satisfy the conditions

$$a \equiv 2(mod\, 3),\ b \equiv 0(mod\, 3),\ b > 0 \quad \text{if } 3 \nmid f,$$

$$a \equiv 6(mod\, 9),\ b \equiv 3 \text{ or } 6(mod\, 9),\ b > 0 \quad \text{if } 3|f.$$

Let θ, θ' and θ'' denote the Gaussian periods for a generating cubic character of K taken with a suitable sign. Then $K = \mathbb{Q}(\theta)$ and

$$Irr(\theta, \mathbb{Q}) = \begin{cases} x^3 + x^2 + ((1-f)/3)x + (f(a-3)+1)/27 & \text{if } 3 \nmid f \\ x^3 - (f/3)x - fa/27 & \text{if } 3|f \end{cases}$$

A proof of this formula is given by Mäki in [6, pp. 7-9].

Now, let K be a cyclic cubic field of conductor f, with ring of integers $\mathcal{R}_K$ and discriminant $D_K = f^2$. Our next task is calculating the λ-invariant for all cyclic cubic fields K up to $D_K < 10^7$. Once we obtain all the λ-invariants, we can then locate the zeroes of each p-adic L-function. We stop at the point 10^7 for the value of D_K, since Llorente and Quer [7] terminate at this value. From their table [7], we have 501 such cyclic cubic fields to consider. For the χ-twisted p-adic L-function, there are two non-trivial branches when $p = 5$ and three non-trivial branches when $p = 7$. So, all together, we have 2505 different branches that need to be calculated, and possible zeroes located.

The methods and techniques we used will be covered in the next section.

3.3 The Methods

Let $p > 3$ be a prime number. Henceforth we fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \hat{\overline{\mathbb{Q}}}_p$ of the algebraic numbers into the complex numbers and into the p-adic Tate field, respectively. One writes $\omega : \mathbb{F}_p^\times \to \mu_{p-1}$ for the Teichmüller character modulo p, and $\langle - \rangle$ denotes the projection to the principal local units, so that $x = \omega(x)\langle x \rangle$ for all $x \in \mathbb{Z}_p^\times$.

For a fixed branch $\beta \in \{0, \ldots, p-2\}$, recall from Chapter 2 that the p-adic L-function $L_p(s, \chi\omega^{1+\beta})$ interpolates

$$L_p(1-n, \chi\omega^{1+\beta}) \;=\; \iota_p \circ \iota_\infty^{-1} \left((1 - \chi\omega^{1+\beta-n}(p)p^{n-1}) \cdot \zeta(1-n, \chi\omega^{1+\beta-n}) \right)$$

at every positive integer n.

If $\omega^{1+\beta}(-1) = -1$ then these values above are identically zero. But we are more interested in the odd branches $\beta \in \{1, 3, \ldots, p-2\}$, since they give the non-trivial branches $L_p(s, \chi\omega^{1+\beta})$, in which case, $\omega^{1+\beta}(-1) = +1$. Now there exist mutually inverse transformations $s \mapsto -\frac{\log_p(1+X)}{\log_p(1+p)}$ and $X \mapsto (1+p)^{-s} - 1$. Under the first, the p-adic L-function is transformed into an element $F_{\chi,\beta}(X) \in \mathcal{O}[\![X]\!]$ where $\mathcal{O} = \mathbb{Z}_p[\mu_3]$ is a complete d.v.r. with residue field $\mathbb{F}_q$, so that

$$q = \begin{cases} p & \text{if } p \equiv 1 \bmod 3 \\ p^2 & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

By the Weierstrass preparation theorem [4], there exists the decomposition

$$F_{\chi,\beta}(X) \;=\; p^{\mu} \times \mathcal{U}(X) \times \left(X^{\lambda} + b_{\lambda-1}X^{\lambda-1} + \cdots + b_0\right)$$

where $\mathcal{U}(X) \in \mathcal{O}[\![X]\!]^{\times}$ is an invertible power series, and each $\left|b_j\right|_p < 1$ in the range $0 \leq j < \lambda$. The invariant λ tells us the number of zeroes of $F_{\chi,\beta}(X)$ on the open p-adic unit disk, at the same time, the integer μ is actually zero by a deep result in [8].

We focus on the 2-modified p-adic L-function $\left(2\omega^{\beta}(2)\langle 2\rangle^{-s}-1\right)\times L_p(s, \chi\omega^{1+\beta})$, and the power series $\mathcal{F}_{\chi,\beta}(X)$ is actually the transform of it. Then for some $\mathcal{U}^{\dagger}(X) \in \mathcal{O}[\![X]\!]^{\times}$, one has

$$\mathcal{F}_{\chi,\beta}(X) \;=\; \mathcal{U}^{\dagger}(X)\times\left(X^{\lambda}+b_{\lambda-1}X^{\lambda-1}+\cdots+b_0\right)\times \begin{cases} (X + \frac{p}{1+p}) & \text{if } \beta \equiv -1 \bmod p-1 \\ 1 & \text{if } \beta \not\equiv -1 \bmod p-1. \end{cases}$$

The set of zeroes for both power series are identical, except if $\beta \equiv -1 \bmod p-1$ in which case there is one extra zero for $\mathcal{F}_{\chi,\beta}(X)$ at $X = -\frac{p}{1+p}$.

3.3.1 Generating the cubic character

In order to calculate the λ-invariant attached to cyclic cubic number fields, we should construct the cubic character χ of each conductor f.

According to [7], every conductor $f < \sqrt{10^7}$ has the decomposition form $f \;=\; 3^{2e_3} \;\times \prod_{l\,\equiv\,1 \bmod 3} l^{e_l}$ where the exponent $e_l \in \{0, 1\}$ for each prime l. Let $\theta_l : \mathbb{F}_l^{\times} \to \mu_{l-1}$ denote the Teichmüller character modulo l, and write θ_9 for a non-trivial character modulo 9 such that $\theta_9 = \theta_9\big|_{\mathbb{F}_3^{\times}} = \mathbf{1}$. Then the function

$$\chi'(n) \;:=\; \begin{cases} \theta_9(n)^{e_3} \times \prod_l \theta_l(n)^{\frac{(l-1)e_l}{3}} & \text{if } \gcd(n, f) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

yields an even cubic character of conductor f taking values in the Eisenstein integers $\mathbb{Z}\left[e^{2\pi i/3}\right]$. It follows that the field cut out by this character, K' say, is a totally real cyclic extension of $\mathbb{Q}$ with discriminant $D_{K'} = f^2$. Note that

$$\theta_9(n) = \begin{cases} 1 & \text{if } n = 1 \\ \xi_3 & \text{if } n = 2 \\ 0 & \text{if } n = 3 \\ \xi_3^2 & \text{if } n = 4 \\ \xi_3 & \text{if } n = 5 \\ 0 & \text{if } n = 6 \\ \xi_3^2 & \text{if } n = 7 \\ 1 & \text{if } n = 8 \\ 0 & \text{if } n = 9. \end{cases}$$

There are exactly 217 out of the 501 cyclic fields of discriminant $< 10^7$ have no other associated non-conjugate cubic field, 126 pairs of fields containing 2 non-conjugate sharing the same discriminant per pair and 8 groups containing 4 non-conjugate sharing the same discriminant per group.

3.3.2 Computing the cyclotomic λ-invariant of K

Wiles [9] proved that the Iwasawa Main Conjecture is true for any totally real fields. Then, the analytic λ-invariant is equal to the algebraic λ-invariant. In which case, we only need calculate the analytic λ-invariant, since it is easier. Let $\lambda_p(\chi\omega^{1+\beta})$ be the λ-invariant attached to the $\chi\omega^{1+\beta}$-twisted p-adic zeta function. We used two totally different methods to determine λ_5 and λ_7. The first method is faster but does not work if $p|f \times \phi(f)$. While the second method is a little bit slow, but it does work for each case.

First Method: One begins by determining the coefficients of the p-adic power series $\mathcal{F}_{\chi,\beta}(X)$ for a fixed branch $\beta \in \{1, 3, \ldots, p-2\}$. Henceforth assume $p \neq 2$ satisfies $\gcd\big(p, f\phi(f)\big) = 1$. Applying a result of Delbourgo [10, Thm

1], one has the Taylor series expansion

$$\mathcal{F}_{\chi,\beta}(X) \;=\; \sum_{j=0}^{\infty} c_j(\mathcal{F}_{\chi,\beta}) \cdot X^j$$

with each coefficient $c_j(\mathcal{F}_{\chi,\beta}) = \lim_{N\to\infty} c_j^{(N)}(\mathcal{F}_{\chi,\beta})$; here the Cauchy sequence $\left\{c_j^{(N)}(\mathcal{F}_{\chi,\beta})\right\}_{N\geq 1}$ is given by the summation

$$c_j^{(N)}(\mathcal{F}_{\chi,\beta}) \;=\; \sum_{m=1,\, p\nmid m}^{p^N} \begin{pmatrix} \mathcal{L}_N(m) \\ j \end{pmatrix} \omega^{\beta}(m) \times \sum_{x=1}^{2f_\chi} a_x(\chi) \cdot \delta_N(x,m).$$

The generated values of χ allow us to compute $a_x(\chi) := \zeta(0,\chi) + \sum_{j=1}^{x-1} \chi(j) - 2\sum_{j=1}^{\lfloor (x-1)/2 \rfloor} \chi(j)$, whilst the functions $\mathcal{L}_N : \mathbb{Z}_p^{\times} \to \mathbb{Z}_{\geq 0}$ and $\delta_N(-,-) : \mathbb{N}\times\mathbb{N} \to \{0,1\}$ are described in [10, §2].

Proposition 3.3 *If* $\mathrm{ord}_p\big(c_j^{(2)}(\mathcal{F}_{\chi,\beta})\big) = 0$ *for some* $j \leq p$*, then*

$$\lambda_p(\chi\omega^{1+\beta}) \;=\; \min\left\{ j \geq 0 \;\middle|\; \mathrm{ord}_p\big(c_j^{(2)}(\mathcal{F}_{\chi,\beta})\big) = 0 \right\} - \begin{cases} 1 & \textit{if } \beta \equiv -1 \mod p-1 \\ 0 & \textit{if } \beta \not\equiv -1 \mod p-1. \end{cases}$$

The proposition is proved by Delbourgo [11]. And here is an example of the PARI code with the first method to determine the λ-invariant when $p = 7$ and $f = 223$.

```
allocatemem(50000000);
\\ Default parameters
pbase = 7;    \\ the prime p in p-adic, which MUST be == 1(mod 3)
N = 7;        \\ where our approximation cuts out at
acc = N+1;    \\ the number of p-adic places to use
\\ Basic functions
ord_p(n) = valuation(n,pbase) \\ ord_p gives the p-adic valuation
padic(x) = x + O(pbase^(acc)) \\ convert a rational to a p-adic number
pu_pow(x,s) = exp(s * log(x)) \\ computes <x>^s
\\ Cubic field parameters
f = 223;                                                   \\ the conductor of cubic character
D = f^2;                                                       \\ the discriminant of the cubic field
z=teichmuller(padic(lift(znprimroot(pbase))))^((pbase-1)/3);  \\ a third root of unity
sumtrunc = floor(N/eulerphi(2*f))+1;                           \\ the power of p to truncate, i.e. "mathcal{T}" in my paper
if(gcd(pbase,f*eulerphi(f))!=1, print("SHIT!!!!"); break; ,);

\\ Computation of a non-trivial character of conductor = f
ourchicomp(n,lbase) = {if(gcd(n,lbase)==1 , if( Mod(n^((lbase-1)/3),lbase) == Mod(1,lbase) , return(padic(1)) ,  if(Mod(n^((lbase-1)/3),lbase)==znprimroot(lbase)^((lbase-1)/3) , return(z), return(z^2) ) ) , return(padic(0)) );}
ourchinine(n) = {if(gcd(n,3)==1, if(Mod(n,9)==Mod(1,9) || Mod(n,9)==Mod(8,9), return(padic(1)), if(Mod(n,9)==Mod(2,9) || Mod(n,9)==Mod(7,9), return(z), return(z^2) ); ); , return(padic(0)) );}
ourchi(n) = {if(gcd(f,3)==1, blob=padic(1), blob=ourchinine(n));
      for(lbase=5, f , if(isprime(lbase) && gcd(lbase,f)==lbase, blob=blob*ourchicomp(n,lbase), 0 ); );return(blob);}

chiarray = vector(2*f);
chivalues = vector(f);
for(i=1, f, chiarray[i]=ourchi(i); chiarray[i+f]=chiarray[i]; chivalues[i]=chiarray[i]; );

\\ Print out the values of chi
\\ print(chivalues);

\\ The a_m(chi) coefficient vector
Aconst = -sum(j=1, f, j*chiarray[j] , 0)/f;
A(m)= {return(Aconst + sum(j=1, m-1, chiarray[j], 0) - (2*sum(j=1, floor((m-1)/2), chiarray[j] , 0)) )}

a = vector(2*f);
a[1] = Aconst;
a[2] = Aconst+chiarray[1];
for(m=3, 2*f, a[m]=a[m-1] + chiarray[m-1] - 2*(floor((m-1)/2)-floor((m-2)/2))*chiarray[floor((m-1)/2)] );
```

```
\\ Algorithm which works out p-adic zeta function with twist omega^(1+beta), and with an extra Euler factor at 2

q = floor(pbase^(sumtrunc*eulerphi(2*f)) / (2*f*pbase^N));  \\ This is "gamma_{N,t}" in my paper
u = lift(Mod(1,2*f)/Mod(pbase,2*f));                         \\ This is "varpi" in my paper

th(x, m) = lift( Mod(m,2*f*(pbase^N))+Mod((x-m)*((pbase*u)^N)-1 , 2*f*(pbase^N)) )+1;

mom(m) = sum(x=1, 2*f, if(th(x,m)< (pbase^(sumtrunc*eulerphi(2*f))-2*f*(pbase^N)*q), a[x] , padic(0) ) );

zetap(s, beta) = {return((sum(m=1, pbase^N, padic(if(gcd(pbase,m)==1, mom(m)*(teichmuller(padic(m))^beta) * pu_pow(padic(m),-s), padic(0) )) , 0)));}

\\ Truncation of log(m)/log(1+p) modulo p^N
lambdaN(m) = lift(Mod((log(m + O(pbase^(N+1)))/log(pbase+1 + O(pbase^(N+1)))), pbase^N))


\\ Work out the j-th coefficient of the polynomial approximation modulo J_N
coeffappr(j, beta) = sum(m=1, pbase^N, if( gcd(pbase,m)==1 && j<=lambdaN(m) , padic(binomial(lambdaN(m),j)) * teichmuller(padic(m))^(beta) * mom(m), padic(0) ) );


\\ The interpolation polynomials F_N^(beta) -- might take ages to run!
polyzeta(beta) = sum(m=1, pbase^N, if(gcd(pbase,m)==1, mom(m) * teichmuller(padic(m))^(beta) * (1+X)^(lambdaN(m)), 0));

\\ Returns the lambda-invariant associated to the beta-branch (without extra Euler factor at p)
laminv(beta) = {
for(i=0, 100, if(valuation(coeffappr(i, beta),pbase)==0, return(i-valuation(gcd(pbase,2^(beta+1)-1),pbase) ); break));
}


\\print(laminv(1););
\\print(laminv(3););
\\print(laminv(5););
\\print("c_0 = "coeffappr(0,1));
\\print("c_1 = "coeffappr(1,1));
\\print("c_2 = "coeffappr(2,1));
\\print("c_3 = "coeffappr(3,1));
\\print("c_4 = "coeffappr(4,1));
\\print("c_5 = "coeffappr(5,1));
\\print("c_6 = "coeffappr(6,1));
```

Second Method: Assume $p \neq 2$ is a prime, and let $B_{n,\chi\omega^{\beta-1}}$ be the $\chi\omega^{\beta-1}$-twisted Bernoulli number of index n. Then the identity

$$\Omega_{\chi,\beta}(r) := L_p\big(-p^r, \chi\omega^{1+\beta}\big) \;=\; \iota_p \circ \iota_\infty^{-1}\left(-\big(1-\chi\omega^{\beta-1}(p)p^{p^r}\big)\cdot\frac{B_{1+p^r,\chi\omega^{\beta-1}}}{1+p^r}\right).$$

follows easily from the interpolation formula of $L_p(s, \chi\omega^{1+\beta})$. It is not hard to see that $\Omega_{\chi,\beta}(r)$ is depending on $B_{1+p^r,\chi\omega^{\beta-1}}$, which can be calculated using the well known formula

$$B_{n,\chi\omega^{\beta-1}} \;=\; \big(f_{\chi\omega^{\beta-1}}\big)^{n-1} \times \sum_{a=1}^{f_{\chi\omega^{\beta-1}}} \chi\omega^{\beta-1}(a) \times \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) B_i \left(\frac{a}{f_{\chi\omega^{\beta-1}}}\right)^{n-i}$$

where $f_{\chi\omega^{\beta-1}}$ is the conductor of $\chi\omega^{\beta-1}$ viewed as a primitive Dirichlet character.

Proposition 3.4 *Expanding* $F_{\chi,\beta}(X) = \sum_{j=0}^{\infty} c_j(F_{\chi,\beta}) \cdot X^j$, *then for all integers* $t \geq 1$*:*

$$c_0(F_{\chi,\beta}) \equiv \Omega_{\chi,\beta}(t-1) \mod p^t, \quad c_1(F_{\chi,\beta}) \equiv \frac{\Omega_{\chi,\beta}(t-1)-\Omega_{\chi,\beta}(2t-1)}{(1+p)^{p^{t-1}}-1} \mod p^t, \quad \textit{and}$$

$$c_2(F_{\chi,\beta}) \;\equiv\; \frac{\Omega_{\chi,\beta}(t-1)-\Omega_{\chi,\beta}(3t-1)}{\big((1+p)^{p^{t-1}}-1\big)^2} + \frac{\Omega_{\chi,\beta}(4t-1)-\Omega_{\chi,\beta}(2t-1)}{\big((1+p)^{p^{2t-1}}-1\big)\big((1+p)^{p^{t-1}}-1\big)} \mod p^t.$$

In particular, choosing $t = 1$ allows us to determine $c_0(F_{\chi,\beta}), c_1(F_{\chi,\beta})$ and $c_2(F_{\chi,\beta})$ modulo p from the terms $\Omega_{\chi,\beta}(0), \ldots, \Omega_{\chi,\beta}(3)$, so one can check whether $\lambda_p(\chi\omega^{1+\beta})$ is 0, 1, 2 or ≥ 3. Of course if the λ-invariant is ≥ 3, then one requires more of these coefficients $c_j(F_{\chi,\beta})$ to calculate it exactly, which may become expensive from a computational perspective.

Again, the above proposition is proved by Delbourgo [11]. Here is the example PARI code we used to determine the λ-invariant when $p = 7$ and $f = 1489$ in the second method.

```
allocatemem(50000000);


\\ Default parameters
pbase = 7;    \\ the prime p in p-adic, which MUST be == 1(mod 3)
acc = 6;    \\ the number of p-adic places to use
\\ Basic functions
ord_p(n) = valuation(n,pbase) \\ ord_p gives the p-adic valuation
padic(x) = x + O(pbase^(acc)) \\ convert a rational to a p-adic number
\\ Cubic field parameters
f = 1489;                                                     \\ the conductor of cubic character
D = f^2;                                                      \\ the discriminant of the cubic field
z=teichmuller(padic(lift(znprimroot(pbase))))^((pbase-1)/3);  \\ a third root of unity
\\ Computation of a non-trivial character of conductor = f
ourchicomp(n,lbase) = {
   if(gcd(n,lbase)==1 , if( Mod(n^((lbase-1)/3),lbase) == Mod(1,lbase) , return(padic(1)) ,  if(Mod(n^((lbase-1)/3),lbase)==znprimroot(lbase)^((lbase-1)/3) , return(z), return(z^2) ) ) , return(padic(0)) );
}
ourchinine(n) = { if(gcd(n,3)==1, if(Mod(n,9)==Mod(1,9) || Mod(n,9)==Mod(8,9), return(padic(1)), if(Mod(n,9)==Mod(2,9) || Mod(n,9)==Mod(7,9), return(z), return(z^2) ); ); , return(padic(0)) );}
ourchi(n) = {if(gcd(f,3)==1, blob=padic(1), blob=ourchinine(n));
      for(lbase=5, f , if(isprime(lbase) && gcd(lbase,f)==lbase, blob=blob*ourchicomp(n,lbase), 0 ); );
  return(blob);}
chivalues = vector(f);
for(i=1, f, chivalues[i]=ourchi(i); );
\\ Functions compute the special value L_p(-pbase^r,chi*omega^{1+beta}) using Bernoulli numbers
tchbet(a,beta) = (Mod(beta,pbase-1)!=Mod(1,pbase-1))*( (gcd(a,pbase)==1)*teichmuller(padic(a))^(beta-1) - padic(1)) + padic(1);
zetval(r,beta) = -(1-chivalues[pbase]*tchbet(pbase,beta)*pbase^(pbase^r) ) * sum(j=1, pbase*f, chivalues[lift(Mod(j-1,f))+1]*tchbet(j,beta)*sum(i=0, 1+pbase^r,
binomial(1+pbase^r,i)*padic(bernfrac(i))*padic(j)^(1+pbase^r-i) * (pbase*f)^i  ) )  / padic((1+pbase^r)*(pbase*f)) ;

\\ Returns the first three coefficients associated to the beta-branch (without extra Euler factor at p)
azero = vector(pbase-1);
aone = vector(pbase-1);
atwo = vector(pbase-1);

for(beta=1, pbase-2, if(Mod(beta,2)==Mod(1,2), azero[beta]=zetval(0,beta); , 0; ); );
for(beta=1, pbase-2, if(Mod(beta,2)==Mod(1,2), aone[beta]=(zetval(0,beta)-zetval(1,beta))/(pbase); , 0; ); );
\\ for(beta=1, pbase-2, if(Mod(beta,2)==Mod(1,2), atwo[beta]=(zetval(0,beta)-zetval(2,beta))/((pbase)^2) + ((zetval(3,beta)-zetval(1,beta))/(pbase*((1+pbase)^(pbase)-1)) ); , 0; ); );
for(beta=1, pbase-2, if(Mod(beta,2)==Mod(1,2), print("a_0(",beta,") = ",azero[beta],","); print("a_1(",beta,") = ",aone[beta],","); print("a_2(",beta,") = ",atwo[beta] ); print(" "); , 0; ); );
```

3.3.3 Locating the zeroes of $L_p(s, \chi\omega^j)$

Our second task is determining the zeroes of the associated $\chi\omega^{1+\beta}$-twisted p-adic L-function if $\lambda\big(\chi\omega^{1+\beta}\big) > 0$. Using the Mazur-Mellin transform, we can map such zero s_0 to the value $x_0 = (1+p)^{-s_0} - 1 \in p\mathbb{Z}_p$, which is also a zero of $\mathcal{F}_{\chi,\beta}$. But, it does not mean every zero of $\mathcal{F}_{\chi,\beta}$ is the image of the transform (see [12] or [13] for more details).

Now, our question is how can we locate the zeroes of $\mathcal{F}_{\chi,\beta}$ if $\lambda\big(\chi\omega^{1+\beta}\big) > 0$. The approach of Ellenberg, Jain and Venkatesh [14, Prop 5.3] tells us that if one know the coefficients $c_j(\mathcal{F}_{\chi,\beta})$ of the power series $\mathcal{F}_{\chi,\beta}(X)$ to accuracy $O(p^{K+1-j})$, then for each $k \leq \lfloor K/\lambda \rfloor$ one can compute the coefficients $a_0, a_1, \ldots, a_{K-\lambda k}$ of its distinguished polynomial up to $O(p^k)$. Once this polynomial has been found modulo p^k then providing $\lambda\big(\mathcal{F}_{\chi,\beta}\big) \leq 4$, the location of its zeroes can be established using classical formulae.

We surprisingly realised that he formula given in [10, Thm 1] for the $c_j(\mathcal{F}_{\chi,\beta})$'s works without restricting p. Furthermore, the formulae also work when $p \big| \phi(f)$ with $p \nmid 2f$.

That leaves us to deal with the situation where $p \mid f$. As we consider only $p = 5$ and $p = 7$, we do not need to worry about $p = 5$ since none of the conductors $f < \sqrt{10^7}$ is divisible by 5. So, assume $p = 7$, and put $\pi_7^+ := e^{2\pi i/7} + e^{-2\pi i/7}$ which generates $\mathbb{Q}(\mu_7)^+$. Here the character χ associated to the cubic field K is such that $\chi = \omega^{2m}\widetilde{\chi}$ for some integer $m \not\equiv 0 \pmod 3$ and cubic character $\widetilde{\chi}$ of conductor $f_{\widetilde{\chi}} = f/7$; one then has an isomorphism $K(\pi_7^+) \cong \widetilde{K}(\pi_7^+)$ where $\widetilde{K}$ denotes the real cubic field of discriminant $f_{\widetilde{\chi}}^2 = f^2/49$ cut out by $\widetilde{\chi}$. It follows that $\mathcal{F}_{\chi,\beta}(X) = \mathcal{F}_{\widetilde{\chi},\beta+2m}(X)$, and to work out the latter's zeroes is straightforward (as discussed above) using [10, Thm 1], because the prime 7 does not divide $f_{\widetilde{\chi}}$.

Here is an example PARI code that we used to locate the zero of $\mathcal{F}_{\chi,\beta}$.

```
allocatemem(50000000);
\\ Default parameters
pbase = 7;   \\ the prime p in p-adic
laminv = 1;  \\ This is the no. of zeroes of the 2-modified L-function
CapK = 6;
littleK = floor(CapK/laminv);
acc = CapK+1;      \\ number of p-adic places to use

\\ Basic functions
padic(x) = x + O(pbase^(acc));

\\ Coefficients of f(T), to be inputted by hand, unfortunately

a = [6*7 + 4*7^2 + 2*7^3 + 3*7^4 + 2*7^5 + 4*7^6 + 5*7^7 + O(7^8), 3 + 3*7 + 7^3 + 4*7^4 + 5*7^5 + 3*7^6 + 2*7^7 + O(7^8), 3 + 3*7 + 7^2 + 7^4 + 5*7^5 + 6*7^6 + 7^7 + O(7^8),
3 + 4*7 + 7^2 + 3*7^4 + 4*7^5 + 4*7^6 + 6*7^7 + O(7^8), 6 + 5*7 + 5*7^2 + 6*7^3 + 5*7^5 + 7^7 + O(7^8), 4 + 7 + 7^2 + 2*7^3 + 7^5 + 2*7^6 + 5*7^7 + O(7^8),
6 + 7 + 4*7^2 + 4*7^3 + 7^4 + 6*7^5 + O(7^8)];

\\ Work out the coefficients b_n modulo p
bmodp = vector(CapK-laminv+1);

bmodp[1] = Mod(1,pbase)/a[1+laminv];
for(s=1, CapK-laminv, bmodp[s+1] = Mod(-1,pbase) * sum(i=1, s, a[1+laminv+i] * bmodp[1+s-i], 0) / a[1+laminv] );

\\ Work out the characteristic zero coefficients b_n
bvecp = vector(CapK-laminv+1);

for(i=0, CapK-laminv, bvecp[i+1] = padic(lift(bmodp[i+1])) );

\\ Iteratively compute better and better bvecp's
for(i=1, 30, bvecp[1] = (1/a[1+laminv]) * (1-sum(j=1, laminv, a[1+laminv-j]*bvecp[1+j], 0) );
for(s=1, CapK-2*laminv, bvecp[1+s] = (-1/a[1+laminv]) * (sum(i=0, laminv+s, a[1+i]*bvecp[1+laminv+s-i], 0) - a[1+laminv]*bvecp[1+s]) ); );

\\ Compute c's from the a's and b's
cvecp = vector(1+laminv);

cvecp[1+laminv] = 1 +O(pbase^(littleK));
for(n=0, laminv-1, cvecp[n+1] = sum(i=0, n, a[i+1]*bvecp[n-i+1], 0)+O(pbase^(littleK)) );
\\ print(cvecp);
print(-cvecp[1]);
```

3.3.4 Determining the size of the class group

As we mentioned before, one can write

$$\mathrm{Irr}(\theta,\mathbb{Q}) \;=\; \begin{cases} x^3+x^2+((1-f)/3)x+(f(a-3)+1)/27 & \text{if } 3\nmid f \\ x^3-(f/3)x-fa/27 & \text{if } 3\mid f \end{cases}$$

where $f=(a^2+3b^2)/4$.

In order to compute the class group of $K(\mu_p)$, we need to find an irreducible polynomial of degree $3(p-1)$ whose roots generate this field. Once we have the polynomials, PARI do the rest for us. Note that although $\mathrm{Irr}(\theta,\mathbb{Q})\times(x^p-1)$ generates the number field, it is **not** irreducible.

Definition 3.3 *For each prime p, one defines the monic polynomial $P_p(x)\in\mathbb{Z}[x]$ by*

$$P_p(x) \;:=\; \left(x^p-(\theta)^p\right)\times\left(x^p-(\theta')^p\right)\times\left(x^p-(\theta'')^p\right)$$

whose roots are precisely $\left\{\left(e^{2\pi i/p}\right)^j\cdot\theta,\ \left(e^{2\pi i/p}\right)^j\cdot\theta',\ \left(e^{2\pi i/p}\right)^j\cdot\theta'' \text{ with } j=0,\ldots,p-1\right\}$.

It follows from this description of its roots that the splitting field of $P_p(x)$ is equal to $\mathbb{Q}(\theta,\mu_p)$. We also observe that $\mathrm{Irr}(\theta,\mathbb{Q})=(x-\theta)(x-\theta')(x-\theta'')$ naturally divides into $P_p(x)$.

Conjecture 3.5 *The quotient polynomial $P_p^\dagger(x):=\frac{P_p(x)}{\mathrm{Irr}(\theta,\mathbb{Q})}$ is irreducible over $\mathbb{Q}$.*

For both $p=5$ and $p=7$, PARI verified the irreducibility of $P_p^\dagger(x)$ up to conductor $f<\sqrt{10^7}$. At these small primes the numerator can be explicitly determined as follows.

Lemma 3.6 *(i) If $p=5$, then*

$$P_5(x)=x^{15}-\mathcal{A}_5x^{10}+\mathcal{B}_5x^5-\mathcal{C}_5$$

where $\mathcal{A}_5=A^5-5A^3B+5A^2C+5AB^2-5BC,$

$\mathcal{B}_5 = B^5 - 5AB^3C + 5B^2C^2 + 5A^2BC^2 - 5AC^3,$

$\mathcal{C}_5 = C^5.$

(ii) If $p = 7$, then

$$P_7(x) = x^{21} - \mathcal{A}_7 x^{14} + \mathcal{B}_7 x^7 - \mathcal{C}_7$$

where $\mathcal{A}_7 = A^7 - 7A^5B + 7A^4C + 14A^3B^2 - 21A^2BC - 7AB^3 + 7AC^2 + 7B^2C,$

$\mathcal{B}_7 = B^7 - 7AB^5C + 7B^4C^2 + 14A^2B^3C^2 - 21AB^2C^3 - 7A^3BC^3 + 7BC^4 +$

$7A^2C^4,$

$\mathcal{C}_7 = C^7.$

Proof. The following argument was found by me. For each prime p and scalars $\alpha, \beta, \gamma \in \mathbb{C}$, consider the polynomial expansion

$$\big(x^p - \alpha^p\big)\big(x^p - \beta^p\big)\big(x^p - \gamma^p\big) = x^{3p} - \big(\alpha^p + \beta^p + \gamma^p\big)x^{2p} + \big(\alpha^p\beta^p + \alpha^p\gamma^p + \beta^p\gamma^p\big)x^p - (\alpha\beta\gamma)^p.$$

Taking $p = 5$ and $(\alpha, \beta, \gamma) = (\theta, \theta', \theta'')$, assertion (i) follows from the identities

$$\begin{aligned} \alpha^5 + \beta^5 + \gamma^5 \;&=\; (\alpha+\beta+\gamma)^5 \;-\; 5(\alpha+\beta+\gamma)^3(\alpha\beta+\alpha\gamma+\beta\gamma) \;+\; 5(\alpha+\beta+\gamma)^2(\alpha\beta\gamma) \\ &\quad + 5(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)^2 \;-\; 5(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha\beta\gamma) \end{aligned}$$

and

$$\begin{aligned} \alpha^5\beta^5 + \alpha^5\gamma^5 + \beta^5\gamma^5 \;&=\; (\alpha\beta+\alpha\gamma+\beta\gamma)^5 \;-\; 5(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)^3(\alpha\beta\gamma) \\ &\quad + 5(\alpha\beta+\alpha\gamma+\beta\gamma)^2(\alpha\beta\gamma)^2 + 5(\alpha+\beta+\gamma)^2(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha\beta\gamma)^2 \\ &\quad - 5(\alpha+\beta+\gamma)(\alpha\beta\gamma)^3. \end{aligned}$$

Similarly, if $p = 7$ and $(\alpha, \beta, \gamma) = (\theta, \theta', \theta'')$ again, then (ii) follows from the identities

$$\begin{aligned} \alpha^7 + \beta^7 + \gamma^7 \;&=\; (\alpha+\beta+\gamma)^7 \;-\; 7(\alpha+\beta+\gamma)^5(\alpha\beta+\alpha\gamma+\beta\gamma) \;+\; 7(\alpha+\beta+\gamma)^4(\alpha\beta\gamma) \\ &\quad + 14(\alpha+\beta+\gamma)^3(\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 21(\alpha+\beta+\gamma)^2(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha\beta\gamma) \\ &\quad - 7(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)^3 \;+\; 7(\alpha+\beta+\gamma)(\alpha\beta\gamma)^2 \\ &\quad + 7(\alpha\beta+\alpha\gamma+\beta\gamma)^2(\alpha\beta\gamma) \end{aligned}$$

and

$$\begin{aligned}
\alpha^7\beta^7+\alpha^7\gamma^7+\beta^7\gamma^7 \;=\; & (\alpha\beta+\alpha\gamma+\beta\gamma)^7 \;-\; 7(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)^5(\alpha\beta\gamma) \\
& + 7(\alpha\beta+\alpha\gamma+\beta\gamma)^4(\alpha\beta\gamma)^2 + 14(\alpha+\beta+\gamma)^2(\alpha\beta+\alpha\gamma+\beta\gamma)^3(\alpha\beta\gamma)^2 \\
& - 21(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)^2(\alpha\beta\gamma)^3 \\
& - 7(\alpha+\beta+\gamma)^3(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha\beta\gamma)^3 \\
& + 7(\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha\beta\gamma)^4 \;+\; 7(\alpha+\beta+\gamma)^2(\alpha\beta\gamma)^4.
\end{aligned}$$

These equations can either be checked by hand(!), or by using a symbolic algebra package. □

The coefficients of $P_p(x)$, and hence of $P_p^\dagger(x)$, can now be calculated for both $p = 5$ and 7. Moreover assuming $P_p^\dagger(x)$ is irreducible, there exists an isomorphism $K(\mu_p) \cong \mathbb{Q}[x]/\langle P_p^\dagger(x)\rangle$ of algebraic extensions of degree $3p-3$ over $\mathbb{Q}$. Lastly the PARI/GP [15] command *bnfinit.clgp.no* works out the class number associated to the quotient polynomial $P_p^\dagger(x)$ at both 5 and 7.

Chapter 4

The Results

We spent more than half a year running the PARI programs in Room G.3.12 at University of Waikato. All the results will be shown in this chapter.

4.1 Summary

In general, we only got either 0 or 1 for all $\lambda_5(\chi\omega^{1+\beta})$. On the other hand, when $p = 7$, we found four "special" cases at $f = 547, 549, 2223$ and 2493, where $\lambda_7(\chi\omega^{1+\beta}) = 3$. Also, there were 27 examples with $\lambda_5(\chi\omega^{1+\beta}) = 2$. The details are given blew.

Table I. *The number of cyclic cubic fields K of discriminant $D_K < 10^7$ with prescribed $\lambda_5(\eta)$*

$p = 5$	$\eta = \chi\omega^2$	$\eta = \chi$
$\#K$ with $\lambda_5(\eta) = 0$	478	483
$\#K$ with $\lambda_5(\eta) = 1$	23	18
$\#K$ with $\lambda_5(\eta) = 2$	0	0
$\#K$ with $\lambda_5(\eta) = 3$	0	0
$\#K$ with $\lambda_5(\eta) \neq 0$	23	18

Table II. *The number of cyclic cubic fields K of discriminant $D_K < 10^7$ with prescribed $\lambda_7(\eta)$*

$p = 7$	$\eta = \chi\omega^2$	$\eta = \chi\omega^4$	$\eta = \chi$
$\#K$ with $\lambda_7(\eta) = 0$	433	432	440
$\#K$ with $\lambda_7(\eta) = 1$	58	57	52
$\#K$ with $\lambda_7(\eta) = 2$	8	10	9
$\#K$ with $\lambda_7(\eta) = 3$	2	2	0
$\#K$ with $\lambda_7(\eta) \neq 0$	68	69	61

It is clear that approximately 4% of cubic 5-adic λ-invariants are positive, whilst roughly 13% of 7-adic λ-invariants are positive.

Another observation is that up to discriminant $D_K < 10^7$, the equivalence

"$\lambda_5(\chi\omega^{1+\beta}) \geq 1$ for some $\beta \in \{1, 3\} \iff$ 5 divides the class number of $K(\mu_5)$"

holds for these cyclic cubic fields K. Similarly, the implication

"$\lambda_7(\chi\omega^{1+\beta}) \geq 1$ for some $\beta \in \{1, 3, 5\} \implies$ 7 divides the class number of $K(\mu_7)$"

holds true up to discriminant $D_K < 10^7$; however the reverse implication turns out to be false.

There are some other deep results found by Delbourgo [11]. One of the most interesting discovery states as follows.

Proposition 4.1 *If $p = 5$ or 7, and for every cyclic cubic field K of discriminant $D_K < 10^7$ and conductor $\mathfrak{f}$, each $(\chi\omega^\beta)^{-1}$-eigenspace in $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathfrak{X}_{\infty,K}$ has a monogenic Λ-module structure, i.e. there exists a pseudo-isomorphism*

$$\left(\mathcal{O} \otimes_{\mathbb{Z}_p} \mathfrak{X}_{\infty,K}\right)^{(\chi^{-1}\omega^{-\beta})} \overset{\text{ps}\,\cong}{\longrightarrow} \mathcal{O}[\![X]\!] \Big/ \left(F_{\chi,\beta}(X)\right).$$

The proof can also be found in [11], and it relies on the tables of zeroes I painstakingly compiled.

4.2 Tables

Table 1. *Data for cyclic cubic extensions with one non-conjugate field per discriminant*

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
7	-1	3	1	0	0	0	0	0	1	1
9	-3	3	1	0	0	0	0	2	1	7
13	5	3	1	0	0	0	0	0	4	13
19	-7	3	1	0	0	0	0	0	13	52
31	-4	6	1	0	0	0	0	0	9	259
37	11	3	1	0	0	0	0	0	49	567
43	8	6	1	0	0	0	0	0	37	513
61	-1	9	1	0	0	1	0	0	117	1519
67	5	9	1	0	0	0	0	0	52	2511
73	-7	9	1	0	0	0	0	1	73	2716
79	17	3	1	0	0	0	0	0	169	2997
97	-19	3	1	0	0	0	0	0	109	27664
103	-13	9	1	0	0	0	1	0	169	5047
109	2	12	1	0	0	0	0	0	121	47952
127	20	6	1	0	0	0	0	0	148	15309
139	23	3	1	1	0	0	0	0	975	20425
151	-19	9	1	0	1	1	0	0	675	24381
157	14	12	1	0	0	0	0	0	169	35152
163	-25	3	4	0	0	0	0	0	2704	81648
181	-7	15	1	0	0	0	0	0	468	880896
193	23	9	1	0	0	2	1	0	793	51597
199	11	15	1	0	0	0	0	0	17904	52117
211	-13	15	1	0	0	0	0	0	1332	41553
223	-28	6	1	0	0	1	0	0	541	96348
229	-22	12	1	0	0	1	0	0	1033	56203
241	17	15	1	0	0	1	0	0	1332	1464463
271	29	9	1	0	0	0	0	0	873	72475
277	26	12	4	0	0	0	0	0	2368	269568
283	32	6	1	0	0	0	0	0	577	314431
307	-16	18	1	0	0	0	0	0	481	215275

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
313	35	3	7	0	0	1	0	2	11116	1680700
331	-1	21	1	0	0	0	0	0	7479	234576
337	5	21	1	0	0	1	0	0	3364	1163484
349	-37	3	4	0	0	0	0	0	24832	941200
367	35	9	1	0	1	0	0	0	1300	487024
373	-13	21	1	0	0	0	0	0	2257	151956
379	29	15	1	0	0	0	0	0	2308	326781
397	-34	12	4	0	0	0	0	0	9472	2285776
409	-31	15	1	0	0	0	0	0	4612	297259
421	-19	21	1	0	0	0	0	0	1737	1229904
433	2	24	1	0	0	0	0	0	1813	641173
439	-28	18	1	0	0	0	0	0	1549	585844
457	-10	24	1	0	0	0	1	1	3796	3619728
463	23	21	1	1	0	0	1	0	4225	2206764
487	-25	21	1	0	0	0	0	0	3412	4313088
499	32	18	1	0	0	0	2	0	2197	420147
523	-43	9	1	0	0	0	0	1	7081	1076677
541	29	21	1	0	0	0	1	0	12987	1975428
547	-1	27	4	0	0	3	0	0	15616	6613488
571	-31	21	1	0	0	0	1	0	6921	8779428
577	11	27	1	0	1	0	1	1	6025	11253487
601	26	24	1	0	0	0	0	1	3681	9807616
607	-49	3	4	0	0	0	0	0	30784	5654800
613	47	9	1	0	0	0	0	0	5101	8643024
619	17	27	1	0	0	1	0	0	14367	1809997
631	-43	15	1	0	0	0	0	0	112896	1841157
643	-40	18	1	0	0	0	0	1	3796	3232831
661	-49	9	1	0	0	1	1	0	14823	1178548
673	-37	21	1	0	0	0	0	0	9001	65542932
691	8	30	1	0	0	1	0	0	4356	15729637

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
709	53	3	4	0	0	0	2	0	113664	4363632
727	44	18	1	0	0	0	0	0	8749	2734069
733	50	12	1	0	0	1	0	0	5284	17781904
739	-16	30	1	0	0	0	0	0	5476	1403649
751	41	21	1	0	0	0	0	0	13941	15012864
757	29	27	1	0	0	0	0	0	5941	5002263
769	-49	15	1	0	0	0	0	1	9028	320314176
787	-31	27	1	0	0	0	0	0	12469	18586288
811	56	6	1	0	0	0	0	0	11691	12986688
823	5	33	1	1	0	0	0	0	18100	5290677
829	-7	33	1	0	0	0	0	0	69123	4282972
853	35	27	4	0	0	2	0	0	211264	60632208
859	-13	33	1	0	0	0	0	0	23893	195898816
877	59	3	7	0	0	1	0	0	1676311	945955584
883	47	21	1	0	0	0	0	1	22021	16717428
907	-19	33	1	0	1	0	0	0	23725	77158656
919	-52	18	1	0	0	0	0	0	16239	3995001
937	-61	3	4	0	0	1	0	0	87616	17508400
967	41	27	1	0	0	0	0	0	185731	6720273
991	-61	9	1	0	0	0	1	0	13941	10522953
997	-10	36	1	0	0	0	0	0	8452	46145407
1009	-43	27	4	0	0	0	0	0	169216	22344768
1021	14	36	1	0	0	0	0	0	7569	53655744
1033	53	21	1	0	0	0	0	0	23569	8984196
1039	59	15	1	0	0	0	0	0	25012	24830416
1051	-64	6	1	0	0	0	0	0	11673	53146800
1063	65	3	13	0	0	0	0	0	304564	1833634075
1069	62	12	1	0	0	0	0	0	29487	12924457
1087	-55	21	1	0	0	0	0	0	31252	9664596
1093	-22	36	1	0	0	0	0	0	16081	10610379

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
1117	65	9	1	0	0	1	0	1	322816	4695327
1123	35	33	1	0	0	0	1	0	46324	9385936
1129	-67	3	7	0	0	0	0	1	137011	67548411
1153	-7	39	1	0	0	0	0	0	23569	40538368
1171	11	39	1	0	0	0	0	0	49257	59573475
1201	59	21	1	0	0	0	0	0	18153	5677776
1213	17	39	1	0	0	0	0	1	25693	6022107
1231	-19	39	1	0	0	1	0	0	118287	7287133
1237	41	33	1	0	1	0	2	0	67225	25980976
1249	53	27	1	0	0	0	0	0	21313	54702064
1279	-43	33	1	0	0	1	0	1	136299	25501021
1291	-67	15	1	0	0	0	0	0	111564	550655728
1297	-25	39	1	0	0	1	0	0	78724	6552819
1303	-55	27	1	0	1	0	1	0	45700	18508581
1321	71	9	1	0	0	0	1	0	22689	286267072
1327	-4	42	1	0	0	0	0	0	11737	22496292
1381	-31	39	1	0	0	0	0	1	118287	185939712
1399	68	18	4	0	0	0	0	0	117136	96987904
1423	20	42	1	0	0	0	0	0	16516	107324352
1429	71	15	1	0	0	0	0	0	188944	23950269
1447	35	39	1	0	0	2	0	0	126736	14510356
1453	-67	21	1	0	0	0	0	0	41257	28816236
1459	56	30	13	0	1	0	2	0	219700	370216756
1471	-76	6	1	1	0	0	0	0	49275	13223763
1483	-37	39	1	0	1	0	1	0	35425	9762025
1489	77	3	19	0	0	1	0	0	1407919	317701888
1531	-7	45	1	0	0	0	0	0	22932	38591296
1543	77	9	1	0	0	0	1	1	50137	37374064
1549	11	45	1	0	0	0	0	0	62308	15739353

f	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
1567	-79	3	7	0	0	1	0	1	216643	96666661
1579	32	42	1	0	0	0	1	0	173559	87526656
1597	50	36	1	0	0	0	0	0	18628	108020304
1609	-19	45	1	0	0	0	0	0	49108	27088033
1621	-79	9	1	0	0	0	0	0	50121	208090701
1627	80	6	1	0	0	0	0	0	230416	26320021
1657	-70	24	1	0	0	0	2	0	90688	192129196
1663	-73	21	1	0	0	0	0	1	56641	135136512
1669	-67	27	1	0	0	0	0	0	35113	45926629
1693	47	39	1	0	0	0	1	0	122581	18537925
1699	-64	30	4	0	0	0	0	0	333552	178204672
1723	-40	42	1	0	0	0	0	0	127504	64885212
1741	-49	39	1	0	0	0	0	1	44937	18478096
1747	-61	33	1	0	0	0	0	0	99733	559892736
1753	-10	48	1	0	0	0	0	0	23236	21948775
1759	-31	45	1	0	0	0	0	1	44452	253908837
1777	14	48	16	0	0	0	0	1	472384	15652623168
1783	83	9	1	0	0	1	1	0	46213	2585548063
1789	-82	12	4	0	0	0	0	0	400192	91108800
1801	74	24	1	0	0	1	0	0	32409	44940987
1831	68	30	7	0	0	0	1	0	145404	768588912
1861	-37	45	1	0	0	0	0	0	45396	101871952
1867	-85	9	1	0	0	0	0	0	152464	31302817
1873	65	33	1	0	0	0	0	0	614656	36498357
1879	-73	27	4	0	0	0	0	1	306496	226343152
1933	62	36	1	0	0	0	0	0	61717	237158064
1951	-1	51	4	0	0	0	0	0	278352	863742208
1987	89	3	7	0	0	0	0	0	460243	607992931
1993	-13	51	1	0	0	1	0	1	88153	46690189
1999	-52	42	1	0	0	0	0	0	129747	35713548

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2749	14	60	1	1	0	0	0	0	486300	496224064
2767	-76	42	1	0	0	0	0	0	87061	206981892
2791	92	30	1	0	0	0	0	0	176724	427928464
2797	89	33	4	0	0	0	0	0	883264	549690624
2803	95	27	4	1	0	1	0	0	307600	1522183936
2833	98	24	1	0	0	0	1	0	156253	538854400
2851	-73	45	1	0	0	0	0	0	388944	171980091
2857	41	57	1	0	0	0	0	0	172993	120466521
2887	-91	33	1	0	0	0	0	0	233809	378993664
2917	-106	12	1	0	0	0	0	0	105337	5304014800
2953	-70	48	1	0	0	0	0	0	86596	809373936
2971	56	54	1	0	0	0	0	0	57573	375031852
3001	77	45	1	0	0	0	0	1	638604	161648116
3019	-13	63	1	0	0	0	0	0	74533	558073152
3037	-49	57	4	0	0	0	1	0	400192	4738134576
3049	17	63	1	0	0	1	0	0	119989	1454096448
3061	38	60	1	0	0	0	0	0	278784	1645822800
3067	-19	63	1	0	0	0	0	1	153733	4638983076
3079	-79	45	1	0	0	0	0	0	395728	403030147
3109	23	63	1	0	0	0	0	0	138613	61825545216
3121	89	39	1	0	0	0	0	0	440559	230734525
2011	59	39	1	0	1	0	0	0	108225	643186575
2017	-34	48	1	0	0	0	0	0	46909	368269200
2029	77	27	1	0	0	0	0	0	114673	420209244
2053	83	21	1	0	0	0	1	0	116473	71838900
2083	23	51	1	0	1	0	0	0	101425	75125961
2089	38	48	1	0	0	0	0	0	280063	53190511
2113	-82	24	1	0	0	0	0	0	335167	64723333
2131	-91	9	4	0	0	2	0	0	1300032	1093485568

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2137	-85	21	1	0	0	0	0	0	90868	34495632
2143	92	6	1	0	0	0	0	0	48217	73583019
2161	29	51	1	0	0	0	0	0	232011	97053775
2179	-88	18	1	0	1	0	0	0	111925	276970752
2203	8	54	1	0	0	0	0	2	35197	41287057
2221	53	45	1	0	0	0	0	0	243756	115809993
2239	-91	15	1	0	0	0	1	0	925104	504712656
2251	-16	54	1	0	0	0	0	0	142587	1061711469
2269	83	27	1	0	0	0	1	0	87313	373996224
2281	86	24	1	0	0	0	0	0	130923	59842273
2287	20	54	1	0	0	0	0	1	49972	2255430625
2293	-37	51	1	0	0	0	0	0	67357	106998975
2311	89	21	4	0	0	1	0	0	353088	7745455872
2341	74	36	1	0	0	0	0	0	28629	136624201
2347	-64	42	1	0	0	0	0	2	312403	41034924
2371	41	51	1	0	0	0	0	0	289683	110257693
2377	-79	33	1	0	0	0	1	0	155281	1977851925
2383	-28	54	1	0	0	0	0	0	65461	107405872
2389	59	45	1	0	0	0	0	0	169068	164226933
2437	-1	57	7	0	0	0	0	0	1217671	622509867
2467	11	57	1	0	0	0	0	0	182293	345349936
2473	-73	39	1	0	0	0	0	0	66193	68167575
2503	47	51	1	1	0	0	0	1	88225	163216053
2521	-97	15	1	0	0	0	0	0	769932	709474896
2539	83	33	1	1	0	0	0	0	235225	168367381
2551	-49	51	1	0	0	0	0	0	117081	120641200
2557	101	3	7	0	0	0	0	0	938119	713161827
2593	-25	57	1	0	0	0	0	0	354064	126118447
2617	-91	27	1	0	0	0	0	0	217921	531443952
2647	29	57	1	0	0	0	0	0	119413	179427339

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2659	-103	3	19	0	0	0	0	0	42893127	1692396139
2671	44	54	1	0	0	0	0	0	91593	76333671
2677	-31	57	1	0	0	0	0	1	119569	659839600
2683	-97	21	1	0	0	0	1	0	100693	99572004
2689	62	48	4	0	0	0	0	0	974784	347380272
2707	-55	51	1	1	0	1	0	0	229300	661949869
2713	-103	9	1	0	0	0	1	0	109129	138680829
2719	101	15	1	0	0	0	0	0	1020688	141136189
2731	104	6	1	0	0	1	0	0	75501	73475451

Table 2. *Data for cyclic cubic extensions with two non-conjugate field per discriminant*

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
63	15	3	3	0	0	0	2	0	444	7
	-12	6	3	0	0	2	0	0	147	7
91	-16	6	3	0	0	0	0	0	219	13
	11	9	3	0	0	0	0	0	543	13
117	-21	3	3	1	0	0	0	0	975	26832
	6	12	3	0	0	0	1	0	471	36309
133	17	9	3	0	0	0	0	0	543	52
	-10	12	3	0	0	0	0	0	444	52
171	24	6	3	0	0	0	0	0	579	62643
	-3	15	3	0	0	0	0	0	1452	670800
217	29	3	3	0	0	0	0	0	2619	259
	-25	9	3	0	0	0	0	0	1404	259
247	-31	3	3	0	0	0	0	0	3099	436449
	-4	18	3	0	0	0	0	0	2019	183675
259	-19	15	3	0	0	0	0	0	2028	567
	8	18	3	0	0	0	0	0	1443	567
279	33	3	3	0	0	0	0	0	6507	491619
	-21	15	3	0	0	0	0	0	3996	296112

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
301	-31	9	3	0	0	0	0	0	5583	513
	23	15	3	0	0	0	0	0	4476	513
333	-30	12	3	0	0	0	0	1	3324	287469
	-3	21	3	0	0	0	0	0	4863	823284
387	-39	3	3	0	0	0	0	0	7527	835029
	15	21	3	0	0	0	0	0	28416	8817984
403	-37	9	3	0	0	0	0	1	8451	1563051
	17	21	3	0	0	0	0	0	12987	1252524
427	41	3	3	1	0	1	0	0	8775	1519
	-40	6	3	0	0	0	0	1	31536	1519
469	-43	3	3	0	0	0	0	0	11199	2511
	38	12	3	0	0	0	0	0	11271	2511
481	41	9	3	0	0	1	1	0	12459	1583631
	14	24	3	0	0	0	1	0	6771	2531088
511	44	6	3	0	0	0	1	0	8859	2716
	-37	15	3	0	0	1	0	0	19452	2716
549	42	12	3	0	0	0	0	0	9747	4454652
	-39	15	3	0	0	3	0	0	11772	2779329
553	-22	24	3	0	0	0	0	0	25887	2997
	5	27	3	0	0	0	0	0	10092	2997
559	47	3	3	0	0	0	0	0	12207	163004400
	-7	27	3	0	0	1	0	0	15483	31414068
589	41	15	3	0	0	0	0	0	296784	13132539
	-13	27	3	0	0	0	0	0	35073	8288253
603	-48	6	3	0	0	0	0	0	13611	14677200
	33	21	3	0	0	0	0	0	21243	4044492
657	51	3	9	0	0	1	0	0	171081	11233089
	-30	24	9	0	0	0	0	0	35892	28105227
679	23	27	3	0	0	0	0	0	21423	27664
	-4	30	12	0	0	0	0	0	108336	27664
703	-52	6	3	1	0	0	0	0	31575	10983843
	-25	27	12	0	0	0	0	0	166512	100158768
711	-39	21	3	0	0	0	1	0	32259	21575484
	-12	30	12	0	0	0	0	0	68016	394849728
721	47	15	3	0	0	1	0	0	32412	5047
	-34	24	3	0	0	0	0	1	18759	5047

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
763	-55	3	12	0	0	0	0	0	337584	47952
	53	9	3	0	0	0	0	0	32079	47952
793	-43	21	3	0	0	0	0	0	27567	15590172
	38	24	3	0	0	0	0	0	16227	9365403
817	-55	9	3	0	0	0	0	0	42492	8738847
	-1	33	3	0	0	0	0	0	62319	299306259
871	53	15	3	0	0	0	0	0	272064	12872925
	-28	30	3	0	0	0	1	0	23916	56173824
873	42	24	3	0	0	0	0	2	13467	29235108
	15	33	3	0	0	0	0	0	63804	6562101
889	-37	27	3	1	0	0	0	0	43275	15309
	17	33	3	0	0	0	0	0	63111	15309
927	60	6	3	0	0	0	0	0	20172	127767792
	-21	33	3	0	0	0	0	0	72327	63216192
949	-58	12	12	0	0	1	0	0	92928	359853312
	23	33	3	0	0	2	0	0	76431	12367047
973	-25	33	3	0	0	0	0	0	1108224	20425
	2	36	3	0	0	0	0	0	61893	20425
981	-57	15	3	0	0	0	0	0	66684	13024557
	51	21	3	0	0	0	1	0	33519	26624052
1027	56	18	3	0	0	1	0	0	14979	17717616
	29	33	3	0	0	0	0	0	84099	39646179
1057	50	24	3	0	0	0	0	1	107892	24381
	-31	33	3	0	0	0	1	0	374301	24381
1099	-61	15	3	0	0	0	0	0	65676	35152
	47	27	3	0	0	0	0	0	85251	35152
1141	26	36	3	0	0	0	0	0	43887	81648
	-1	39	3	0	0	0	0	0	51123	81648
1143	-57	21	3	0	0	0	0	1	54147	86526468
	-3	39	12	0	0	0	0	0	320304	343901376
1147	-49	27	12	0	0	0	1	0	145584	165935952
	5	39	3	0	0	1	0	0	85644	79606800
1159	44	30	3	0	0	0	0	0	132516	54580071
	-37	33	3	0	0	1	1	0	1373679	19576137
1251	-48	30	3	0	0	0	0	0	309348	65948007
	-21	39	3	1	0	0	0	0	270225	18862032

f	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
1261	-61	21	3	0	0	0	0	0	152967	461314224
	-34	36	21	0	0	0	0	0	217749	3028530897
1267	71	3	9	0	0	0	0	0	272241	880896
	-64	18	36	0	0	0	0	0	623376	880896
1273	-58	24	3	0	0	0	1	0	272181	55879551
	23	39	3	0	0	0	0	0	111567	21988827
1333	-70	12	12	0	0	0	0	0	207792	181503504
	38	36	3	0	0	0	0	0	34047	27605367
1339	-73	3	9	0	0	0	0	0	408357	145199691
	8	42	9	0	1	1	1	0	178425	143393796
1351	-52	30	3	0	0	1	0	2	45084	51597
	29	39	3	0	0	0	2	1	2929593	51597
1359	69	15	3	0	0	0	0	0	1131408	85492017
	-12	42	3	0	0	0	0	0	190593	410670000
1387	65	21	3	0	0	0	0	0	304896	1134820800
	-16	42	3	0	1	0	1	1	59475	190519056
1393	-73	9	3	0	0	0	0	0	262989	52117
	62	24	3	0	0	0	0	0	143109	52117
1413	-75	3	12	0	0	0	0	0	438384	309270000
	33	39	3	0	0	0	0	0	1771329	124283712
1417	59	27	3	0	0	0	0	0	59151	66056877
	-49	33	3	0	0	0	1	0	201099	53261712
1467	51	33	3	1	0	0	0	0	164775	124054173
	24	42	3	0	0	0	0	0	54399	70142436
1477	74	12	3	0	0	0	0	0	55107	41553
	-61	27	3	0	0	0	0	0	122499	41553
1501	-73	15	3	0	0	0	0	0	359892	40090869
	-46	36	3	0	0	0	0	0	92781	427147344
1561	41	39	9	0	0	0	0	1	665937	96348
	-13	45	9	0	0	0	1	0	371124	96348
1591	71	21	3	0	0	0	0	0	226551	537897024
	17	45	3	0	1	0	0	0	144300	279006525
1603	65	27	3	0	0	0	0	1	77628	56203
	-43	39	3	0	0	0	1	0	162147	56203
1629	78	12	3	0	0	0	0	0	72927	312113763
	-57	33	3	1	0	1	0	0	219375	1235558961

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
1651	77	15	3	0	0	0	0	0	311628	890803008
	23	45	3	0	0	0	0	1	103692	961716483
1687	-76	18	39	0	0	0	0	1	653211	1464463
	59	33	3	1	0	0	1	0	370575	1464463
1737	-75	21	3	0	0	0	1	0	733872	87574284
	6	48	3	0	0	0	0	0	88491	162596889
1791	-84	6	3	0	0	0	0	0	279621	168482352
	51	39	3	0	0	0	0	0	503721	55885089
1807	71	27	3	0	0	0	0	0	1211787	1241589897
	44	42	3	0	0	0	0	0	381357	1395997200
1843	80	18	3	0	0	0	0	0	76044	546100464
	53	39	3	0	0	0	0	0	196851	110188911
1891	83	15	3	0	0	1	0	0	492804	73960887
	-79	21	3	0	0	0	0	0	551853	795425088
1897	-55	39	3	0	0	0	0	0	7115472	72475
	26	48	3	0	0	0	0	0	109863	72475
1899	87	3	12	0	0	0	0	0	783216	1220537808
	-48	42	12	0	0	1	0	0	430704	463769712
1939	67	33	3	0	0	0	0	0	400791	269568
	41	45	3	0	0	0	0	0	227964	269568
1957	86	12	12	0	0	0	0	1	451776	1859280192
	5	51	3	0	0	0	0	0	239916	887058225
1963	-88	6	3	0	0	0	0	0	2038527	1138178223
	-7	51	3	0	0	0	0	0	538569	118069812
1981	-43	45	3	0	0	0	0	0	178428	314431
	11	51	3	0	0	0	0	0	358059	314431
2007	69	33	3	0	0	0	0	0	392907	684211437
	15	51	3	0	0	0	1	0	1014528	5167773387
2041	89	9	3	0	0	0	0	0	256503	2989233408
	-19	51	3	0	0	0	0	0	176943	8969052144
2061	87	15	3	0	0	1	0	1	245388	193018791
	-21	51	3	0	0	0	0	0	178743	232546548
2071	-61	39	3	0	0	0	0	0	571869	130842621
	47	45	3	0	0	0	0	0	505044	188171667
2077	-91	3	12	0	0	0	1	0	835056	355849200
	71	33	3	0	0	0	0	0	2728593	698409216

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2119	-76	30	21	0	1	0	2	1	909300	3344878719
	-49	45	3	0	0	0	0	1	324732	698151636
2149	89	15	3	0	0	0	0	0	1162416	215275
	-73	33	3	1	0	0	0	0	562575	215275
2169	-93	3	12	0	0	0	1	0	1083888	1050302064
	42	48	12	0	0	1	1	0	3048192	607890864
2191	-31	51	3	0	0	0	2	1	675471	1680700
	-4	54	3	0	0	2	1	0	97599	1680700
2257	-46	48	3	0	0	0	0	0	191187	322382025
	35	51	3	0	0	0	0	0	461484	485424576
2263	95	3	21	0	0	0	0	2	2813076	1436703177
	-67	39	3	0	0	0	0	0	333099	503559408
2317	95	9	3	0	0	0	0	0	548532	234576
	-94	12	3	0	0	0	0	0	722277	234576
2353	-85	27	12	0	0	0	0	0	2478384	5675065200
	50	48	3	0	0	0	0	0	219132	2920268403
2359	-97	3	9	1	0	0	0	1	950625	1163484
	92	18	9	1	0	0	1	0	378225	1163484
2413	-97	9	3	0	0	0	0	0	176907	1738116144
	-43	51	3	0	0	1	0	0	288543	150862257
2439	-84	30	9	0	0	0	0	1	653508	883903356
	-3	57	9	0	0	1	1	0	724221	1308574449
2443	32	54	3	0	0	0	0	0	126219	941200
	5	57	3	0	0	0	0	0	4323072	941200
2449	-61	45	3	0	0	0	0	1	5957712	359936703
	-7	57	3	0	0	0	0	0	746577	175686084
2479	68	42	3	0	0	0	0	0	200019	971316684
	-13	57	3	0	0	0	0	1	278499	843633063
2493	-93	21	3	0	0	1	3	0	500799	1795385088
	15	57	3	0	0	1	0	0	187356	426758787
2509	98	12	3	0	0	0	0	0	257727	2297775924
	17	57	3	0	0	0	0	0	260247	2351267757
2547	-75	39	3	0	0	0	0	0	1120944	613476675
	-21	57	3	0	0	0	1	0	685659	7842078972
2569	-58	48	3	0	0	0	0	0	156099	487024
	23	57	3	0	0	0	0	0	560271	487024

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2587	95	21	12	0	0	0	0	2	3774096	1279072704
	-40	54	3	0	0	0	0	0	317988	392699847
2611	101	9	3	0	0	0	0	0	289227	151956
	-88	30	3	0	0	0	0	0	421356	151956
2623	-85	33	3	0	0	0	0	0	738828	562223349
	77	39	3	1	0	1	0	0	753975	179115300
2641	-94	24	3	0	0	0	0	0	744237	604114329
	-67	45	3	0	0	0	0	0	3145104	233300919
2653	-82	36	3	0	0	0	0	0	105591	326781
	53	51	3	0	0	0	0	0	782499	326781
2701	-79	39	3	0	0	0	1	0	321087	2636293149
	2	60	3	0	0	0	0	0	4205568	478307349
2743	-100	18	12	0	0	0	0	0	679536	19345303872
	35	57	3	0	0	0	0	0	230796	4049880768
2763	105	3	63	0	0	0	0	0	5180364	92665483092
	-57	51	9	0	0	1	0	0	1874457	1505915775
2779	71	45	3	0	0	0	0	0	756444	2285776
	-37	57	3	0	0	0	0	0	301143	2285776
2817	-66	48	48	0	0	0	0	0	19870464	9059359296
	-39	57	3	1	0	1	0	0	328575	2875158741
2863	83	39	3	0	0	0	0	0	15182157	297259
	-52	54	3	0	0	0	0	0	165567	297259
2869	107	3	9	0	0	0	0	1	4275099	1922172525
	26	60	9	0	0	0	0	0	2135484	2393907696
2881	101	21	12	0	0	0	0	0	1191936	28706317056
	-61	51	3	0	0	0	0	0	592851	2433468339
2899	-97	27	3	0	0	0	0	0	365403	4270431321
	-43	57	3	0	0	1	0	0	6997107	8688086757
2923	107	9	3	0	0	0	0	0	372783	212044959
	80	42	21	0	0	0	0	0	1270164	3373062588
2947	104	18	9	0	0	0	0	0	606609	1229904
	-85	39	9	0	0	0	0	0	1773252	1229904
2977	-109	3	9	0	0	0	0	0	1720017	1155795381
	-1	63	36	0	0	0	0	0	8269056	9758016144
2979	96	30	3	0	0	0	0	0	540756	819232947
	-93	33	3	0	0	0	0	0	3881277	948154896

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
2983	-103	21	3	0	0	0	0	1	539247	6968759616
	5	63	3	0	0	0	0	0	313932	2965310316
3007	65	51	3	0	1	0	2	0	2084400	2547479025
	11	63	3	0	0	0	0	0	647379	5902011648
3031	-109	9	3	0	0	0	0	0	873111	641173
	107	15	3	0	0	0	0	0	337116	641173
3033	-102	24	3	0	0	2	0	1	285051	13114048311
	87	39	3	0	0	0	0	0	595911	1263382029
3073	95	33	3	0	0	0	0	0	16843008	585844
	-67	51	3	0	0	0	0	0	695379	585844
3097	101	27	3	0	0	1	0	0	487479	2865788667
	74	48	3	0	0	0	0	0	277203	1768754475
3133	110	12	3	0	0	0	0	0	357804	11845898688
	-25	63	3	0	0	0	0	0	1076976	2404920000
3139	-109	15	3	0	0	0	0	0	465708	1681338141
	53	57	3	0	0	0	1	1	584607	8528079168
3141	69	51	3	0	0	1	0	0	2845101	4899643329
	42	60	3	0	1	1	1	0	237900	4608277716

Table 3. *Data for cyclic cubic extensions with four non-conjugate field per discriminant*

$\mathfrak{f}$	a	b	$\mathbf{h}(K)$	$\lambda_5(\chi\omega^2)$	$\lambda_5(\chi)$	$\lambda_7(\chi\omega^2)$	$\lambda_7(\chi\omega^4)$	$\lambda_7(\chi)$	$\mathbf{h}_5$	$\mathbf{h}_7$
819	-57	3	9	0	0	1	0	0	502047	36309
	51	15	9	0	0	0	0	1	361296	36309
	24	30	9	0	0	0	0	0	41508	26832
	-3	33	9	0	0	0	0	0	247221	26832
1197	69	3	9	0	0	0	0	0	258021	670800
	-66	12	36	0	0	0	0	0	553536	670800
	-39	33	9	0	0	0	0	0	335997	62643
	15	39	9	0	0	0	0	0	225108	62643
1729	83	3	9	0	0	0	0	0	546813	183675
	-79	15	9	0	0	0	0	0	724932	183675
	29	45	9	0	0	0	0	0	309348	436449
	2	48	9	0	0	0	0	0	2565927	436449
1953	78	24	9	0	0	0	0	0	703809	296112
	-57	39	9	0	0	0	0	0	506493	296112
	-30	48	9	0	0	0	0	0	256932	491619
	-3	51	9	0	0	0	0	0	4804191	491619
2223	87	21	9	0	0	0	0	0	514737	7246008576
	-75	33	9	0	0	0	0	0	9604476	2013977709
	60	42	9	0	0	0	0	0	1379088	2358338112
	33	51	9	0	0	0	3	2	1640637	18440478063
2331	96	6	9	0	0	0	0	0	521001	823284
	-93	15	9	0	0	0	0	0	3707136	823284
	69	39	9	0	0	0	1	0	1673253	287469
	-39	51	9	0	0	1	0	0	928161	287469
2709	-102	12	36	0	1	0	0	0	1040400	8817984
	87	33	9	0	0	0	0	0	1634373	8817984
	33	57	9	0	0	0	0	0	1879641	835029
	6	60	9	0	0	0	0	0	2106576	835029
2821	-103	15	9	1	0	0	0	0	31622400	1252524
	86	36	9	0	0	0	0	0	632853	1252524
	59	51	9	0	0	0	1	0	1621377	1563051
	-22	60	9	0	0	1	0	0	1053972	1563051

Tabulated below are the locations for the zeroes of $\mathcal{F}_{\chi,\beta}(X)$ arising from positive $\lambda\big(\chi\omega^{1+\beta}\big)$'s. As previously mentioned, if $\beta \equiv -1 \bmod p-1$ then there is always a trivial zero at $X = -\frac{p}{p+1}$ for the p-adic power series $\mathcal{F}_{\chi,\beta}(X)$, which we have not bothered to write down in these tables. The rows consist of the coefficients of the power series $\mathcal{F}_{\chi,\beta}$, followed by the zeroes $x_1, \ldots, x_\lambda$. Note that if $p = 5$ we have written $\xi_3 \in \mathbb{Q}_5(\mu_3)$ for the primitive third root of unity $\iota_5\big(e^{2\pi i/3}\big)$, whilst if $p = 7$ we have written ξ_3 to denote the primitive third root of unity $\omega(2) \in \mathbb{Q}_7$.

Table 4. *The zeroes associated to each branch $\mathcal{F}_{\chi,\beta}(X)$ at $p = 5$, with conductor $\mathfrak{f} = \frac{a^2+3b^2}{4}$*

$\mathfrak{f} = 117 = \frac{(-21)^2+3\times 3^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5 + 4\times 5^2 + 4\times 5^4 + 5^5 + 5^6 + O(5^7))\xi_3 + (2\times 5 + O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4 + 2\times 5 + 2\times 5^2 + 4\times 5^3 + 3\times 5^4 + 4\times 5^5 + O(5^6))\xi_3 + (3 + 5 + 2\times 5^2 + 5^3 + 2\times 5^4 + 5^5 + O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3\times 5 + O(5^5))\xi_3 + (3 + 4\times 5^2 + 5^3 + O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1 + 5 + 2\times 5^2 + 3\times 5^3 + O(5^4))\xi_3 + (2 + 5 + 4\times 5^2 + 5^3 + O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1 + 5 + 3\times 5^2 + O(5^3))\xi_3 + (3 + 4\times 5 + 3\times 5^2 + O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1 + 4\times 5 + O(5^2))\xi_3 + (2 + 5 + O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3 + O(5^1))\xi_3 + (1 + O(5^1))$
x_1	$(4\times 5 + 2\times 5^2 + 5^3 + 5^4 + 2\times 5^5 + O(5^6))\xi_3 + (3\times 5 + 3\times 5^3 + O(5^6))$

$\mathfrak{f} = 139 = \frac{23^2+3\times 3^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5 + O(5^7))\xi_3 + (3\times 5^2 + 2\times 5^4 + 3\times 5^5 + O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(5^2 + 3\times 5^3 + 4\times 5^4 + O(5^6))\xi_3 + (1 + 5 + 5^2 + 5^3 + 3\times 5^4 + 5^5 + O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4 + 5 + 4\times 5^2 + 3\times 5^3 + 3\times 5^4 + O(5^5))\xi_3 + (2\times 5 + 4\times 5^2 + 3\times 5^3 + O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4 + 2\times 5 + 4\times 5^2 + 3\times 5^3 + O(5^4))\xi_3 + (1 + 2\times 5 + 3\times 5^3 + O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1 + 5^2 + O(5^3))\xi_3 + (2 + 5 + O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2 + 5 + O(5^2))\xi_3 + (3 + 3\times 5 + O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4 + O(5^1))\xi_3 + (O(5^1))$
x_1	$(5 + 3\times 5^2 + 3\times 5^3 + 3\times 5^4 + 3\times 5^5 + O(5^6))\xi_3 + (3\times 5^2 + 5^3 + 4\times 5^4 + 3\times 5^5 + O(5^6))$

$\mathfrak{f} = 151 = \frac{(-19)^2+3\times 9^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5^2 + 2\times 5^4 + 3\times 5^5 + O(5^7))\xi_3 + (5^3 + 3\times 5^4 + 3\times 5^5 + 5^6 + O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2\times 5 + 4\times 5^2 + 4\times 5^3 + 4\times 5^4 + 5^5 + O(5^6))\xi_3 + (4\times 5 + 5^2 + 5^3 + 5^4 + 2\times 5^5 + O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4 + 2\times 5 + 4\times 5^2 + 4\times 5^3 + O(5^5))\xi_3 + (4 + 2\times 5 + 5^3 + 2\times 5^4 + O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3 + 5 + 2\times 5^2 + 5^3 + O(5^4))\xi_3 + (3 + 2\times 5 + 5^3 + O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2 + 3\times 5 + O(5^3))\xi_3 + (2 + 5 + 2\times 5^2 + O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4 + O(5^2))\xi_3 + (1 + 2\times 5 + O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3 + O(5^1))\xi_3 + (3 + O(5^1))$
x_1	$(3\times 5 + 4\times 5^2 + O(5^3))\xi_3 + (3\times 5 + 3\times 5^2 + O(5^3))$

$\mathfrak{f} = 367 = \frac{35^2+3\times 9^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5^2+2\times 5^4+2\times 5^6+O(5^7))\xi_3+(3\times 5^2+3\times 5^3+3\times 5^4+4\times 5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+3\times 5^2+3\times 5^4+2\times 5^5+O(5^6))\xi_3+(2\times 5+3\times 5^2+2\times 5^3+3\times 5^4+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3\times 5+3\times 5^2+5^3+O(5^5))\xi_3+(4+3\times 5+5^2+3\times 5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+5+O(5^4))\xi_3+(3+3\times 5+4\times 5^2+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+4\times 5^2+O(5^3))\xi_3+(1+5+4\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+O(5^2))\xi_3+(3+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(4\times 5+3\times 5^2+O(5^3))\xi_3+(3\times 5+O(5^3))$

$\mathfrak{f} = 427 = \frac{41^2+3\times 3^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^3+5^4+2\times 5^6+O(5^7))\xi_3+(3\times 5+4\times 5^2+4\times 5^3+4\times 5^4+4\times 5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+5^2+4\times 5^3+2\times 5^4+O(5^6))\xi_3+(3+3\times 5+2\times 5^3+5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+4\times 5^3+O(5^5))\xi_3+(3+3\times 5+4\times 5^3+2\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+3\times 5^2+O(5^4))\xi_3+(2+4\times 5+5^2+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5^2+O(5^3))\xi_3+(4+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4\times 5+O(5^2))\xi_3+(2+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(3+O(5^1))$
x_1	$(2\times 5+4\times 5^2+5^3+5^4+O(5^6))\xi_3+(2\times 5+4\times 5^2+2\times 5^3+5^5+O(5^6))$

$\mathfrak{f} = 463 = \frac{23^2+3\times 21^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$O(5^7)\xi_3+(4\times 5+5^3+5^4+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+5^2+4\times 5^3+5^5+O(5^6))\xi_3+(2+4\times 5+4\times 5^2+5^3+3\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+2\times 5+4\times 5^4+O(5^5))\xi_3+(2+5+3\times 5^2+5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4\times 5+4\times 5^2+4\times 5^3+O(5^4))\xi_3+(3\times 5+4\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5+O(5^3))\xi_3+(4\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5+O(5^2))\xi_3+(4+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(2+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(3\times 5+2\times 5^2+4\times 5^3+2\times 5^4+2\times 5^5+O(5^6))\xi_3+(4\times 5+3\times 5^3+4\times 5^4+4\times 5^5+O(5^6))$

$\mathfrak{f} = 577 = \frac{11^2+3\times 27^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5^2+4\times 5^3+3\times 5^4+5^5+O(5^7))\xi_3+(2\times 5^3+4\times 5^4+3\times 5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2\times 5+4\times 5^3+3\times 5^4+2\times 5^5+O(5^6))\xi_3+(5+5^2+4\times 5^3+3\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5^3+2\times 5^4+O(5^5))\xi_3+(1+2\times 5+2\times 5^2+2\times 5^3+2\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+5^2+4\times 5^3+O(5^4))\xi_3+(2+5^2+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+5+O(5^3))\xi_3+(3+4\times 5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(O(5^2))\xi_3+(5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(3\times 5+4\times 5^2+O(5^3))\xi_3+(4\times 5+4\times 5^2+O(5^3))$

$\mathfrak{f} = 703 = \frac{(-52)^2+3\times 6^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+4\times 5^2+4\times 5^4+5^5+5^6+O(5^7))\xi_3+(5^3+5^4+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(5+4\times 5^2+5^3+2\times 5^5+O(5^6))\xi_3+(4+3\times 5+3\times 5^2+2\times 5^3+2\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+O(5^5))\xi_3+(4+4\times 5+4\times 5^2+4\times 5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+4\times 5^2+O(5^4))\xi_3+(2\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+2\times 5^2+O(5^3))\xi_3+(2+2\times 5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+5+O(5^2))\xi_3+(2+2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(4\times 5+2\times 5^2+4\times 5^3+O(5^6))\xi_3+(2\times 5^2+3\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$

$\mathfrak{f} = 823 = \frac{5^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5^2+3\times 5^3+3\times 5^4+4\times 5^5+2\times 5^6+O(5^7))\xi_3+(3\times 5+2\times 5^2+2\times 5^3+2\times 5^4+4\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+2\times 5^2+3\times 5^3+4\times 5^5+O(5^6))\xi_3+(3+5+2\times 5^2+3\times 5^3+3\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+5+2\times 5^2+2\times 5^3+2\times 5^4+O(5^5))\xi_3+(3+3\times 5^2+5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+5^2+4\times 5^3+O(5^4))\xi_3+(4+3\times 5+3\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+3\times 5+4\times 5^2++O(5^3))\xi_3+(4+2\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+O(5^2))\xi_3+(3+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(2+O(5^1))$
x_1	$(4\times 5+2\times 5^3+4\times 5^4+3\times 5^5+O(5^6))\xi_3+(5+4\times 5^2+2\times 5^3+3\times 5^4+5^5+O(5^6))$

$\mathfrak{f} = 889 = \frac{(-37)^2+3\times 27^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5+5^2+5^3+4\times 5^4+5^5+5^6+O(5^7))\xi_3+(5+3\times 5^4+5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+5^2+4\times 5^3+4\times 5^4+O(5^6))\xi_3+(2+2\times 5+4\times 5^2+2\times 5^3+3\times 5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+5^2+4\times 5^3+2\times 5^4+O(5^5))\xi_3+(2+4\times 5+3\times 5^2+2\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5^2+O(5^4))\xi_3+(4+3\times 5+4\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+2\times 5^2+O(5^3))\xi_3+(4+3\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+5+O(5^2))\xi_3+(3+2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(2+O(5^1))$
x_1	$(2\times 5^2+4\times 5^3+3\times 5^4+4\times 5^5+O(5^6))\xi_3+(2\times 5+5^3+2\times 5^4+5^5+O(5^6))$

$\mathfrak{f} = 907 = \frac{(-19)^2+3\times 33^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5^2+2\times 5^5+5^6+O(5^7))\xi_3+(3\times 5^3+2\times 5^4+2\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+4\times 5^4+2\times 5^5+O(5^6))\xi_3+(3\times 5+2\times 5^2+2\times 5^3+3\times 5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+2\times 5^3+3\times 5^4+O(5^5))\xi_3+(3+5+4\times 5^2+4\times 5^3+5^4+3\times 5^5+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3\times 5+5^3+O(5^4))\xi_3+(4+3\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+O(5^3))\xi_3+(4+5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+O(5^2))\xi_3+(2+3\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(3\times 5+O(5^3))\xi_3+(3\times 5+O(5^3))$

$\mathfrak{f}=1237=\frac{41^2+3\times 33^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5^2+3\times 5^5+4\times 5^6+O(5^7))\xi_3+(2\times 5^3+3\times 5^4+3\times 5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3\times 5+2\times 5^2+4\times 5^4+3\times 5^5+O(5^6))\xi_3+(5+3\times 5^3+3\times 5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+4\times 5^2+4\times 5^3+3\times 5^4+O(5^5))\xi_3+(1+4\times 5+4\times 5^2+3\times 5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5+4\times 5^2+O(5^4))\xi_3+(4\times 5+3\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+5^2+O(5^3))\xi_3+(5+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+O(5^2))\xi_3+(1+2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(3+O(5^1))$
x_1	$(2\times 5+2\times 5^2+O(5^3))\xi_3+(3\times 5+3\times 5^2+O(5^3))$

$\mathfrak{f}=1251=\frac{(-21)^2+3\times 39^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+4\times 5^3+3\times 5^5+3\times 5^6+O(5^7))\xi_3+(3\times 5+4\times 5^2+2\times 5^3+5^4+5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+2\times 5^2+4\times 5^3+2\times 5^5+O(5^6))\xi_3+(4+3\times 5+2\times 5^2+5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+5^2+5^3+3\times 5^4+O(5^5))\xi_3+(2+4\times 5+2\times 5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+5^3+O(5^4))\xi_3+(4+3\times 5+5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+5+5^2+O(5^3))\xi_3+(2+2\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+O(5^2))\xi_3+(2+5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(2+O(5^1))$
x_1	$(4\times 5+3\times 5^2+4\times 5^3+2\times 5^4+3\times 5^5+O(5^6))\xi_3+(4\times 5+5^2+3\times 5^3+5^4+3\times 5^5+O(5^6))$

$\mathfrak{f}=1303=\frac{(-55)^2+3\times 27^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5^2+4\times 5^3+5^4+4\times 5^5+2\times 5^6+O(5^7))\xi_3+(4\times 5^2+3\times 5^4+5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+4\times 5^2+3\times 5^3+5^4+5^5+O(5^6))\xi_3+(5+5^2+5^3+3\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3\times 5^3+2\times 5^4+O(5^5))\xi_3+(2+3\times 5+3\times 5^2+3\times 5^3+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+5^2+5^3+O(5^4))\xi_3+(4\times 5+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3\times 5+O(5^3))\xi_3+(4+5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+O(5^2))\xi_3+(3+2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(2+O(5^1))$
x_1	$(3\times 5+5^2+O(5^3))\xi_3+(3\times 5+2\times 5^2+O(5^3))$

$\mathfrak{f}=1339=\frac{8^2+3\times 42^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5^3+5^6+O(5^7))\xi_3+(3\times 5^2+4\times 5^3+5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+2\times 5^2+4\times 5^3+5^4+2\times 5^5+O(5^6))\xi_3+(2\times 5+4\times 5^2+4\times 5^3+2\times 5^4O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^2+3\times 5^3+O(5^5))\xi_3+(4+5+5^2+4\times 5^3+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+5^2+O(5^4))\xi_3+(1+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+5+O(5^3))\xi_3+(4+4\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(5+O(5^2))\xi_3+(O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(3+O(5^1))$
x_1	$(2\times 5+3\times 5^2+O(5^3))\xi_3+(2\times 5^2+O(5^3))$

$\mathfrak{f} = 1387 = \frac{(-16)^2+3\times 42^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+2\times 5^3+4\times 5^4+3\times 5^5+3\times 5^6+O(5^7))\xi_3+(5^2+2\times 5^4+3\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+2\times 5^3+2\times 5^4+2\times 5^5+O(5^6))\xi_3+(4\times 5^2+2\times 5^3+4\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5^3+2\times 5^4+O(5^5))\xi_3+(4+4\times 5+3\times 5^2+4\times 5^3+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+3\times 5+3\times 5^2+O(5^4))\xi_3+(3+3\times 5+5^2+5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+O(5^3))\xi_3+(4\times 5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3\times 5+O(5^2))\xi_3+(3+3\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(2+O(5^1))$
x_1	$(2\times 5+O(5^3))\xi_3+(2\times 5+2\times 5^2+O(5^3))$

$\mathfrak{f} = 1459 = \frac{56^2+3\times 30^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+4\times 5^3+O(5^7))\xi_3+(4\times 5^2+3\times 5^3+2\times 5^4+5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3\times 5+2\times 5^2+4\times 5^3+4\times 5^4+2\times 5^5+O(5^6))\xi_3+(2\times 5+5^2+3\times 5^3+3\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+3\times 5^2+2\times 5^3+5^4+O(5^5))\xi_3+(3+3\times 5^2+2\times 5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+2\times 5^3+O(5^4))\xi_3+(4+2\times 5+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+2\times 5^2+O(5^3))\xi_3+(1+2\times 5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3+5+O(5^2))\xi_3+(O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(4\times 5+O(5^3))\xi_3+O(5^3)$

$\mathfrak{f} = 1467 = \frac{51^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5+5^2+2\times 5^3+2\times 5^4+4\times 5^6+O(5^7))\xi_3+(5+2\times 5^2+3\times 5^3+5^4+2\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^2+3\times 5^3+3\times 5^5+O(5^6))\xi_3+(3+3\times 5+3\times 5^2+2\times 5^3+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+5^2+5^4+O(5^5))\xi_3+(3+4\times 5+5^2+3\times 5^3+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+2\times 5+3\times 5^3+O(5^4))\xi_3+(2+2\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5^2+O(5^3))\xi_3+(4+5+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+O(5^2))\xi_3+(4+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(O(5^1))$
x_1	$(5^2+2\times 5^3+4\times 5^4+O(5^6))\xi_3+(3\times 5+3\times 5^2+2\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$

$\mathfrak{f} = 1471 = \frac{(-76)^2+3\times 6^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+5^2+2\times 5^3+2\times 5^5+3\times 5^6+O(5^7))\xi_3+(2\times 5+2\times 5^3+3\times 5^4+3\times 5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2+5+2\times 5^2+5^3+2\times 5^5+O(5^6))\xi_3+(3+5+4\times 5^2+3\times 5^3+4\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+5+2\times 5^2+5^4+O(5^5))\xi_3+(3+4\times 5+5^2+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+4\times 5^2+4\times 5^3+O(5^4))\xi_3+(2\times 5+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5^2+O(5^3))\xi_3+(4\times 5+4\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+5+O(5^2))\xi_3+(4+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(4+O(5^1))$
x_1	$(5+5^2+5^3+2\times 5^4+2\times 5^5+O(5^6))\xi_3+(2\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$

$\mathfrak{f} = 1483 = \frac{(-37)^2+3\times 39^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5^2+5^6+O(5^7))\xi_3+(2\times 5^3+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3\times 5+4\times 5^4+2\times 5^5+O(5^6))\xi_3+(3\times 5+2\times 5^2+3\times 5^4+3\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3\times 5+3\times 5^2+3\times 5^3+2\times 5^4+O(5^5))\xi_3+(3+5+5^3+2\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+2\times 5^3+O(5^4))\xi_3+(3+5+5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+3\times 5^2+O(5^3))\xi_3+(2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+O(5^2))\xi_3+(4+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(2+O(5^1))\xi_3+(O(5^1))$
x_1	$(4\times 5+5^2+O(5^3))\xi_3+(5^2+O(5^3))$

$\mathfrak{f} = 1591 = \frac{17^2+3\times 45^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+5^3+3\times 5^4+4\times 5^6+O(5^7))\xi_3+(5^2+2\times 5^3+5^4+2\times 5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4\times 5+4\times 5^2+3\times 5^3+4\times 5^4+5^5+O(5^6))\xi_3+(3\times 5+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5^4+O(5^5))\xi_3+(2+5+4\times 5^2+4\times 5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+5+5^2+4\times 5^3+O(5^4))\xi_3+(2+4\times 5+4\times 5^2+5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2\times 5+3\times 5^2+O(5^3))\xi_3+(2+2\times 5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4\times 5+O(5^2))\xi_3+(3+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(2\times 5+3\times 5^2+O(5^3))\xi_3+(4\times 5+5^2+O(5^3))$

$\mathfrak{f} = 1629 = \frac{(-57)^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^3+3\times 5^4+3\times 5^5+2\times 5^6+O(5^7))\xi_3+(3\times 5^2+5^3+4\times 5^4+5^5+5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^2+4\times 5^3+3\times 5^4+O(5^6))\xi_3+(2\times 5+4\times 5^2+3\times 5^3+3\times 5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+5+5^2+2\times 5^3+O(5^5))\xi_3+(3\times 5^2+5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+2\times 5^2+3\times 5^3+O(5^4))\xi_3+(2\times 5^2+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+5^2+O(5^3))\xi_3+(4+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+5+O(5^2))\xi_3+(2+3\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(O(5^1))$
x_1	$(2\times 5^2+2\times 5^3+2\times 5^4+O(5^6))\xi_3+(2\times 5^2+3\times 5^3+5^4+O(5^6))$

$\mathfrak{f} = 1687 = \frac{59^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5+4\times 5^2+5^3+2\times 5^4+4\times 5^6+O(5^7))\xi_3+(2\times 5+4\times 5^2+5^3+2\times 5^4+4\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+3\times 5^2+2\times 5^3+4\times 5^4+3\times 5^5+O(5^6))\xi_3+(2+4\times 5+5^3+5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5^2+2\times 5^3+O(5^5))\xi_3+(5+5^2+5^3+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5^2+3\times 5^3+O(5^4))\xi_3+(4+3\times 5+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3\times 5+2\times 5^2+O(5^3))\xi_3+(4+5+O(5^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+3\times 5+O(5^2))\xi_3+(2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(2+O(5^1)$
x_1	$(2\times 5+4\times 5^3+5^4+4\times 5^5+O(5^6))\xi_3+(3\times 5^3+5^4+4\times 5^5+O(5^6))$

$\mathfrak{f}=2011=\frac{59^2+3\times 39^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5^3+3\times 5^4+4\times 5^5+5^6+O(5^7))\xi_3+(2\times 5^2+2\times 5^4+3\times 5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3\times 5+2\times 5^2+5^3+4\times 5^4+5^5+O(5^6))\xi_3+(2\times 5+2\times 5^2+4\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+3\times 5^2+2\times 5^3+5^4+O(5^5))\xi_3+(4\times 5+5^2+4\times 5^3+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+3\times 5^2+O(5^4))\xi_3+(4+2\times 5+3\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^2+O(5^3))\xi_3+(4\times 5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+O(5^2))\xi_3+(2+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(4\times 5+2\times 5^2+O(5^3))\xi_3+(4\times 5+O(5^3))$

$\mathfrak{f}=2083=\frac{23^2+3\times 51^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+2\times 5^4+5^5+4\times 5^6+O(5^7))\xi_3+(2\times 5^2+2\times 5^3+3\times 5^4+2\times 5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2\times 5+2\times 5^2+4\times 5^3+2\times 5^4+4\times 5^5+O(5^6))\xi_3+(4\times 5+5^2+2\times 5^3+3\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+2\times 5^3+5^4+O(5^5))\xi_3+(2+3\times 5^2+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2\times 5+4\times 5^3+O(5^4))\xi_3+(1+5^2+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5+2\times 5^2+O(5^3))\xi_3+(4\times 5+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+O(5^2))\xi_3+(4+4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(4\times 5+2\times 5^2+O(5^3))\xi_3+(4\times 5+4\times 5^2+O(5^3))$

$\mathfrak{f}=2119=\frac{(-76)^2+3\times 30^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+3\times 5^3+4\times 5^5+O(5^7))\xi_3+(4\times 5^2+4\times 5^3+2\times 5^4+5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(5^2+5^3+5^5+O(5^6))\xi_3+(2\times 5+5^2+5^3+4\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+5+2\times 5^3+5^4+O(5^5))\xi_3+(3+5+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+2\times 5^3+O(5^4))\xi_3+(1+2\times 5+4\times 5^2+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(5+3\times 5^2+O(5^3))\xi_3+(2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3+5+O(5^2))\xi_3+(3\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(O(5^1))$
x_1	$(4\times 5+2\times 5^2+O(5^3))\xi_3+(5+4\times 5^2+O(5^3))$

$\mathfrak{f}=2149=\frac{(-73)^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5+4\times 5^2+5^3+5^5+O(5^7))\xi_3+(3\times 5+5^3+5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+3\times 5^2+4\times 5^3+O(5^6))\xi_3+(3+3\times 5+3\times 5^2+2\times 5^3+2\times 5^4+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+5+5^2+5^3+5^4+O(5^5))\xi_3+(3+2\times 5+2\times 5^2+5^4++O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+3\times 5+5^3+O(5^4))\xi_3+(4+4\times 5+3\times 5^2+2\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1+5+4\times 5^2+O(5^3))\xi_3+(4+3\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+O(5^2))\xi_3+(3+5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(5^2+2\times 5^3+O(5^6))\xi_3+(4\times 5+3\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$

$\mathfrak{f} = 2179 = \frac{(-88)^2+3\times 18^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+3\times 5^3+5^4+4\times 5^5+5^6+O(5^7))\xi_3+(5^2+4\times 5^3+5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2\times 5^2+2\times 5^3+5^5+O(5^6))\xi_3+(5+2\times 5^2+4\times 5^3+5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+3\times 5^2+5^3+2\times 5^4+O(5^5))\xi_3+(3\times 5+2\times 5^2+5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(5+3\times 5^2+3\times 5^3+O(5^4))\xi_3+(1+3\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1+O(5^3))\xi_3+(3+4\times 5+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+O(5^2))\xi_3+(2+2\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(2+O(5^1))$
x_1	$(2\times 5+4\times 5^2+O(5^3))\xi_3+(3\times 5+O(5^3))$

$\mathfrak{f} = 2359 = \frac{(-97)^2+3\times 3^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5+4\times 5^2+2\times 5^3+5^4+5^5+2\times 5^6+O(5^7))\xi_3+(5+4\times 5^2+2\times 5^3+5^4+5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+2\times 5^2+2\times 5^3+3\times 5^4+4\times 5^5+O(5^6))\xi_3+(3+5^2+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+3\times 5^2+4\times 5^3+O(5^5))\xi_3+(3+3\times 5+5^2+5^3+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+4\times 5+5^2+2\times 5^3+O(5^4))\xi_3+(4+4\times 5+4\times 5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^2+O(5^3))\xi_3+(1+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2\times 5+O(5^2))\xi_3+(1+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(1+O(5^1))$
x_1	$(3\times 5^2+2\times 5^3+2\times 5^4+2\times 5^5+O(5^6))\xi_3+(3\times 5+2\times 5^2+5^3+5^5+O(5^6))$

$\mathfrak{f} = 2359 = \frac{92^2+3\times 18^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5^2+3\times 5^3+5^4+2\times 5^5+O(5^7))\xi_3+(3\times 5^2+3\times 5^3+2\times 5^4+5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2+3\times 5+4\times 5^2+5^3+3\times 5^4+4\times 5^5+O(5^6))\xi_3+(2\times 5+3\times 5^2+2\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+5+5^2+2\times 5^4+O(5^5))\xi_3+(4+2\times 5+2\times 5^2+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+3\times 5^3+O(5^4))\xi_3+(1+3\times 5+3\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+2\times 5^2+O(5^3))\xi_3+(5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2\times 5+O(5^2))\xi_3+(4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(4\times 5^2+5^3+2\times 5^4+2\times 5^5+O(5^6))\xi_3+(5^2+3\times 5^3+5^4+4\times 5^5+O(5^6))$

$\mathfrak{f} = 2503 = \frac{47^2+3\times 51^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5+2\times 5^2+3\times 5^3+5^4+5^5+2\times 5^6+O(5^7))\xi_3+(4\times 5^2+5^3+3\times 5^4+2\times 5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+3\times 5^2+4\times 5^3+4\times 5^5+O(5^6))\xi_3+(1+2\times 5+4\times 5^2+2\times 5^3+5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+5+2\times 5^2+2\times 5^3+3\times 5^4+O(5^5))\xi_3+(4\times 5+4\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5^3+O(5^4))\xi_3+(1+2\times 5+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3\times 5+4\times 5^2+O(5^3))\xi_3+(2+2\times 5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+O(5^2))\xi_3+(3+5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(3+O(5^1))$
x_1	$(5+2\times 5^2+5^3+5^4+5^5+O(5^6))\xi_3+(4\times 5+4\times 5^4+O(5^6))$

$\mathfrak{f}=2539=\frac{83^2+3\times 33^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5+5^2+2\times 5^3+5^4+2\times 5^6+O(5^7))\xi_3+(5+3\times 5^2+3\times 5^3+4\times 5^4+3\times 5^5+4\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+5+5^2+4\times 5^3+5^4+5^5+O(5^6))\xi_3+(3+5+4\times 5^2+2\times 5^3+2\times 5^4+3\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+5+4\times 5^2+4\times 5^3+2\times 5^4+O(5^5))\xi_3+(3+4\times 5+3\times 5^2+2\times 5^3+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(5+5^3+O(5^4))\xi_3+(3+3\times 5+5^2+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2+5+5^2+O(5^3))\xi_3+(1+2\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+O(5^2))\xi_3+(O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(O(5^1))$
x_1	$(2\times 5^2+2\times 5^4+3\times 5^5+O(5^6))\xi_3+(3\times 5+4\times 5^2+2\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$

$\mathfrak{f}=2623=\frac{77^2+3\times 39^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5+4\times 5^3+3\times 5^5+3\times 5^6+O(5^7))\xi_3+(3\times 5+4\times 5^2+4\times 5^4+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+3\times 5^4+O(5^6))\xi_3+(3+3\times 5+5^2+3\times 5^3+5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+5+5^2+4\times 5^3+3\times 5^4+O(5^5))\xi_3+(5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2\times 5^3+O(5^4))\xi_3+(2+5^2+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+5+5^2+O(5^3))\xi_3+(2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+5+O(5^2))\xi_3+(3+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(O(5^1))$
x_1	$(3\times 5^2+5^3+4\times 5^4+3\times 5^5+O(5^6))\xi_3+(4\times 5+3\times 5^2+5^3+5^5+O(5^6))$

$\mathfrak{f}=2707=\frac{(-55)^2+3\times 51^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+3\times 5^2+4\times 5^3+3\times 5^5+3\times 5^6+O(5^7))\xi_3+(5+4\times 5^3+3\times 5^4+5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+5^2+5^3+3\times 5^4+3\times 5^5+O(5^6))\xi_3+(3+2\times 5+5^3+2\times 5^4+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+5+4\times 5^2+2\times 5^3+5^4+O(5^5))\xi_3+(4+3\times 5^3+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+3\times 5^2+2\times 5^3+O(5^4))\xi_3+(4+5+3\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+O(5^3))\xi_3+(3+5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+O(5^2))\xi_3+(4\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(O(5^1))$
x_1	$(2\times 5+2\times 5^2+5^3+3\times 5^4+3\times 5^5+O(5^6))\xi_3+(2\times 5+3\times 5^2+3\times 5^3+3\times 5^4+5^5+O(5^6))$

$\mathfrak{f}=2709=\frac{(-102)^2+3\times 12^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$O(5^7)\xi_3+(5^2+5^3+2\times 5^4+3\times 5^5+5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(5+3\times 5^2+5^3+4\times 5^4+3\times 5^5+O(5^6))\xi_3+(5^2+4\times 5^3+2\times 5^4+2\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+5+4\times 5^2+3\times 5^3+3\times 5^4+O(5^5))\xi_3+(4+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+3\times 5^2+2\times 5^3+O(5^4))\xi_3+(3+4\times 5+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5+5^2+O(5^3))\xi_3+(1+4\times 5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(O(5^2))\xi_3+(2\times 5+2\times 5^2+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(2\times 5+4\times 5^2+O(5^3))\xi_3+(4\times 5+3\times 5^2+O(5^3))$

$\mathfrak{f} = 2749 = \frac{14^2+3\times 60^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(4\times 5+4\times 5^2+5^3+2\times 5^5+3\times 5^6+O(5^7))\xi_3+(3\times 5+4\times 5^2+4\times 5^3+2\times 5^4+2\times 5^5+2\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(1+2\times 5+3\times 5^3+5^5+O(5^6))\xi_3+(4+4\times 5+5^2+5^3+4\times 5^4+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+4\times 5^2+3\times 5^4+O(5^5))\xi_3+(3\times 5+2\times 5^2+2\times 5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(3+5+5^3+O(5^4))\xi_3+(3+2\times 5+4\times 5^2+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2\times 5+5^2+O(5^3))\xi_3+(2+2\times 5+2\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(4+2\times 5+O(5^2))\xi_3+(2+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(1+O(5^1))\xi_3+(O(5^1))$
x_1	$(4\times 5+2\times 5^2+O(5^6))\xi_3+(4\times 5+4\times 5^2+5^4+O(5^6))$

$\mathfrak{f} = 2803 = \frac{95^2+3\times 27^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5+2\times 5^3+2\times 5^4+4\times 5^5+O(5^7))\xi_3+(4\times 5+4\times 5^2+5^4+4\times 5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5^2+3\times 5^3+5^4+O(5^6))\xi_3+(1+5+5^2+4\times 5^3+2\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(3+4\times 5^4+O(5^5))\xi_3+(2\times 5^2+5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+5+4\times 5^3+O(5^4))\xi_3+(3+2\times 5+4\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+2\times 5^2+O(5^3))\xi_3+(2+2\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+O(5^2))\xi_3+(2+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(3+O(5^1))\xi_3+(4+O(5^1))$
x_1	$(2\times 5^3+3\times 5^4+5^5+O(5^6))\xi_3+(5+4\times 5^2+3\times 5^3+4\times 5^4+3\times 5^5+O(5^6))$

$\mathfrak{f} = 2817 = \frac{(-39)^2+3\times 57^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times 5^3+5^5+O(5^7))\xi_3+(5+5^3+5^4+4\times 5^5+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3+3\times 5+4\times 5^2+5^3+3\times 5^4+5^5+O(5^6))\xi_3+(1+3\times 5+4\times 5^3+4\times 5^4+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+4\times 5^2+4\times 5^3+4\times 5^4+O(5^5))\xi_3+(2\times 5+4\times 5^2+4\times 5^3+3\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+4\times 5+3\times 5^2+5^3+O(5^4))\xi_3+(4\times 5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4\times 5+O(5^3))\xi_3+(4\times 5+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+3\times 5+O(5^2))\xi_3+(4+3\times 5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(2+O(5^1))\xi_3+(2+O(5^1))$
x_1	$(4\times 5+2\times 5^2+3\times 5^3+2\times 5^4+5^5+O(5^6))\xi_3+(5+3\times 5^3+4\times 5^4+4\times 5^5+O(5^6))$

$\mathfrak{f} = 2821 = \frac{(-103)^2+3\times 15^2}{4},\ p=5,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(3\times 5+2\times 5^2+3\times 5^4+2\times 5^5+4\times 5^6+O(5^7))\xi_3+(3\times 5^2+4\times 5^4+3\times 5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2+2\times 5+5^2+4\times 5^3+2\times 5^5+O(5^6))\xi_3+(4+3\times 5+4\times 5^2+3\times 5^3+3\times 5^4+4\times 5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(4+3\times 5+3\times 5^3+5^4+O(5^5))\xi_3+(2\times 5+3\times 5^2+4\times 5^3+2\times 5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(1+4\times 5+2\times 5^2+3\times 5^3+O(5^4))\xi_3+(3+5+5^2+3\times 5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(3+2\times 5^2+O(5^3))\xi_3+(4+3\times 5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+5+O(5^2))\xi_3+(4+5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(O(5^1))\xi_3+(2+O(5^1))$
x_1	$(4\times 5+5^2+3\times 5^3+4\times 5^4+4\times 5^5+O(5^6))\xi_3+(2\times 5+4\times 5^2+5^3+5^4+2\times 5^5+O(5^6))$

$\mathfrak{f}=3007=\frac{65^2+3\times51^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(5^2+3\times5^3+4\times5^4+2\times5^5+5^6+O(5^7))\xi_3+(3\times5^2+5^3+3\times5^4+4\times5^5+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(2\times5+4\times5^3+4\times5^5+O(5^6))\xi_3+(5+4\times5^2+2\times5^3+4\times5^4+5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+4\times5+2\times5^2+4\times5^3+2\times5^4+O(5^5))\xi_3+(3+5+5^2+5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(2+2\times5+2\times5^2+3\times5^3+O(5^4))\xi_3+(4+2\times5^2+4\times5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(2\times5+4\times5^2+O(5^3))\xi_3+(3+3\times5+3\times5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(3\times5+O(5^2))\xi_3+(4+4\times5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(4+O(5^1))\xi_3+(3+O(5^1))$
x_1	$(2\times5^2+O(5^3))\xi_3+(4\times5+5^2+O(5^3))$

$\mathfrak{f}=3141=\frac{42^2+3\times60^2}{4},\ p=5,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$(2\times5^2+3\times5^3+4\times5^5+O(5^7))\xi_3+(2\times5^2+3\times5^5+4\times5^6+O(5^7))$
$c_1(\mathcal{F}_{\chi,\beta})$	$(3\times5+3\times5^2+4\times5^4+3\times5^5+O(5^6))\xi_3+(3\times5+3\times5^2+2\times5^4+4\times5^5+O(5^6))$
$c_2(\mathcal{F}_{\chi,\beta})$	$(1+4\times5+4\times5^2+4\times5^3+4\times5^4+O(5^5))\xi_3+(1+4\times5+3\times5^2+3\times5^3+5^4+O(5^5))$
$c_3(\mathcal{F}_{\chi,\beta})$	$(4\times5+4\times5^2+5^3+O(5^4))\xi_3+(2+3\times5^2+2\times5^3+O(5^4))$
$c_4(\mathcal{F}_{\chi,\beta})$	$(4+5+3\times5^2+O(5^3))\xi_3+(1+5+5^2+O(5^3))$
$c_5(\mathcal{F}_{\chi,\beta})$	$(2+O(5^2))\xi_3+(4+2\times5+O(5^2))$
$c_6(\mathcal{F}_{\chi,\beta})$	$(2+O(5^1))\xi_3+(1+O(5^1))$
x_1	$(4\times5^2+O(5^3))\xi_3+(3\times5+2\times5^2+O(5^3))$

Table 5. *The zeroes associated to each branch $\mathcal{F}_{\chi,\beta}(X)$ at $p=7$, with conductor $\mathfrak{f}=\frac{a^2+3b^2}{4}$*

$\mathfrak{f}=9=\frac{(-3)^2+3\times3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+4\times7^3+3\times7^4+7^5+4\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times7+2\times7^2+5\times7^3+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6\times7+3\times7^2+5\times7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times7+3\times7^2+2\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times7+3\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=2\times7+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=4\times7+O(7^2)$

$\mathfrak{f}=61=\frac{(-1)^2+3\times9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+2\times7+2\times7^2+7^3+5\times7^4+3\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+7+7^2+4\times7^3+6\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+6\times7+6\times7^2+3\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+2\times7+2\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+2\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$2\times7+4\times7^2+4\times7^3+5\times7^4+5\times7^5+O(7^6)$

$\mathfrak{f}=63=\frac{(-12)^2+3\times6^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=9$

$\mathfrak{f} = 63 = \frac{15^2+3\times 3^2}{4},\ p=7,\ \beta=3$	see $\mathfrak{f}=9$

$\mathfrak{f} = 73 = \frac{(-7)^2+3\times 9^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7^2+2\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+5\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+2\times 7^2+4\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+6\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$7+4\times 7^2+O(7^3)$

$\mathfrak{f} = 103 = \frac{(-13)^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+6\times 7^3+5\times 7^4+2\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+2\times 7^2+6\times 7^3+2\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+4\times 7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+6\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times 7+5\times 7^2+3\times 7^3+6\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 117 = \frac{6^2+3\times 12^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+4\times 7^2+3\times 7^3+4\times 7^4+2\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+7^2+3\times 7^3+7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+6\times 7^2+6\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+6\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$6\times 7+3\times 7^2+5\times 7^3+3\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 151 = \frac{(-19)^2+3\times 9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+6\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+3\times 7^2+3\times 7^3+7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4\times 7+5\times 7^2+6\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$5\times 7+5\times 7^2+5\times 7^3+6\times 7^5+O(7^6)$

$\mathfrak{f} = 193 = \frac{23^2+3\times 9^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7 + 2\times 7^2 + 6\times 7^3 + 7^5 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 5\times 7^2 + 6\times 7^3 + 4\times 7^4 + 6\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1 + 5\times 7 + 5\times 7^2 + 5\times 7^3 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4 + 5\times 7 + 2\times 7^2 + 5\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5 + 6\times 7 + 5\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1 + 4\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4 + O(7^1)$
x_1 x_2	$\mathcal{A} + \sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A} = 4\times 7 + 6\times 7^2 + O(7^3)$ $\mathcal{A} - \sqrt{\Delta}\mathbb{Q}uad$ and $\Delta = 6\times 7 + 3\times 7^2 + O(7^3)$

$\mathfrak{f} = 193 = \frac{23^2+3\times 9^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7 + 3\times 7^2 + 3\times 7^3 + 2\times 7^4 + 4\times 7^5 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1 + 7 + 6\times 7^2 + 7^3 + 7^4 + 4\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4\times 7 + 2\times 7^2 + 6\times 7^3 + 6\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4 + 3\times 7^2 + 5\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6\times 7 + 2\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4 + O(7^1)$
x_1	$6\times 7 + 4\times 7^2 + 4\times 7^3 + 5\times 7^4 + 2\times 7^5 + O(7^6)$

$\mathfrak{f} = 223 = \frac{(-28)^2+3\times 6^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6 + 6\times 7 + 3\times 7^2 + 7^3 + 6\times 7^4 + 3\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4 + 2\times 7 + 4\times 7^2 + 5\times 7^3 + 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 7^2 + 4\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6 + 2\times 7 + 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2 + 2\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1 + O(7^1)$
x_1	$2\times 7 + 2\times 7^2 + 7^3 + 6\times 7^5 + O(7^6)$

$\mathfrak{f} = 229 = \frac{(-22)^2+3\times 12^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3 + 4\times 7 + 6\times 7^2 + 7^4 + 6\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7 + 7^2 + 3\times 7^3 + 2\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6 + 5\times 7 + 6\times 7^2 + 5\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1 + 3\times 7 + 6\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4 + 3\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3 + O(7^1)$
x_1	$4\times 7 + 7^2 + 7^3 + 2\times 7^4 + 7^5 + O(7^6)$

$\mathfrak{f} = 241 = \frac{17^2+3\times15^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7+2\times7^2+5\times7^3+7^4+2\times7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+6\times7+5\times7^2+3\times7^3+3\times7^4+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+5\times7+5\times7^2+6\times7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+4\times7+2\times7^2+3\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+6\times7+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+3\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$5\times7+7^2+6\times7^5+O(7^6)$

$\mathfrak{f} = 313 = \frac{35^2+3\times3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7+7^2+2\times7^3+4\times7^4+3\times7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+2\times7+2\times7^2+4\times7^3+4\times7^4+2\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+6\times7^3+4\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times7+4\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+6\times7+2\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times7+5\times7^2+6\times7^3+3\times7^4+5\times7^5+O(7^6)$

$\mathfrak{f} = 313 = \frac{35^2+3\times3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+6\times7^3+3\times7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times7+7^3+2\times7^4+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5\times7+3\times7^3+3\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+4\times7+2\times7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+5\times7+5\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=5\times7+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=2\times7+O(7^2)$

$\mathfrak{f} = 333 = \frac{(-30)^2+3\times12^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+5\times7^3+4\times7^4+5\times7^5+2\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times7^2+7^4+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+3\times7+5\times7^2+3\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+3\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times7+3\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$7+2\times7^2+O(7^3)$

$\mathfrak{f} = 337 = \frac{5^2+3\times 21^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+3\times 7^2+2\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+6\times 7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5\times 7+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$4\times 7+7^2+3\times 7^3+5\times 7^4+7^5+O(7^6)$

$\mathfrak{f} = 403 = \frac{(-37)^2+3\times 9^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+3\times 7^3+3\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+7^2+2\times 7^3+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+5\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$6\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 427 = \frac{41^2+3\times 3^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=61$

$\mathfrak{f} = 427 = \frac{(-40)^2+3\times 6^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f}=61$

$\mathfrak{f} = 457 = \frac{(-10)^2+3\times 24^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+2\times 7^2+7^3+3\times 7^4+6\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+5\times 7^2+7^3+3\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+5\times 7^2+2\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+5\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+4\times 7^2+6\times 7^3+2\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 457 = \frac{(-10)^2+3\times 24^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+4\times 7^3+5\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+2\times 7^3+5\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+6\times 7^2+4\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+6\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$4\times 7+7^2+O(7^3)$

$\mathfrak{f} = 463 = \frac{23^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+6\times 7^2+6\times 7^3+4\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+7^2+2\times 7^3+6\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+5\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+7+6\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$6\times 7+6\times 7^2+2\times 7^3+4\times 7^4+7^5+O(7^6)$

$\mathfrak{f} = 481 = \frac{41^2+3\times 9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+6\times 7^2+5\times 7^3+6\times 7^4+4\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+4\times 7^2+5\times 7^3+7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+3\times 7^2+3\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+2\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$4\times 7+5\times 7^2+2\times 7^3+5\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 481 = \frac{41^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+3\times 7^3+2\times 7^4+4\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+7^2+7^3+7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+3\times 7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$6\times 7+4\times 7^2+3\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 481 = \frac{14^2+3\times 24^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+3\times 7^3+2\times 7^4+4\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7^3+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+4\times 7^2+7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+4\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$7+6\times 7^2+2\times 7^3+4\times 7^4+O(7^6)$

$\mathfrak{f}=499=\frac{32^2+3\times18^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7+7^2+4\times7^3+3\times7^4+7^5+4\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times7+5\times7^2+7^3+2\times7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+7+7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+2\times7+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times7+2\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=0\times7+3\times7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=6\times7+5\times7^2+O(7^3)$

$\mathfrak{f}=511=\frac{(-37)^2+3\times15^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=73$

$\mathfrak{f}=511=\frac{44^2+3\times6^2}{4},\ p=5,\ \beta=3$	see $\mathfrak{f}=73$

$\mathfrak{f}=523=\frac{(-43)^2+3\times9^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times7^2+7^3+6\times7^4+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times7+7^2+3\times7^3+5\times7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times7+3\times7^2+6\times7^3+4\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+3\times7+3\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$5\times7+5\times7^2+O(7^3)$

$\mathfrak{f}=541=\frac{29^2+3\times21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7+2\times7^4+5\times7^5+2\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+6\times7+7^2+6\times7^3+4\times7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times7^2+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times7+4\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+3\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$4\times7+6\times7^3+6\times7^4+5\times7^5+O(7^6)$

$\mathfrak{f}=547=\frac{(-1)^2+3\times27^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7+4\times7^2+4\times7^3+2\times7^4+3\times7^5+5\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7^2+3\times7^4+3\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6\times7+4\times7^2+7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times7+3\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+5\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1 x_2 x_3	$\mathcal{A}+y+z$ where $y=(\mathcal{B}+\sqrt{\Delta})^{1/3},\ z=(\mathcal{B}-\sqrt{\Delta})^{1/3}$ $\mathcal{A}+\xi_3y+\xi_3^2z$ with $\mathcal{A}=5\times7+O(7^2),\ \mathcal{B}=7+O(7^2)\mathbb{Q}quad$ $\mathcal{A}+\xi_3^2y+\xi_3z$ and $\Delta=7^2+O(7^3)\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad$

$\mathfrak{f} = 549 = \frac{(-39)^2+3\times 15^2}{4},\ p=7,\ \beta=1$		
$c_0(\mathcal{F}_{\chi,\beta})$	$7+7^2+7^3+2\times 7^5+5\times 7^6+O(7^7)$	
$c_1(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+6\times 7^3+3\times 7^4+7^5+O(7^6)$	
$c_2(\mathcal{F}_{\chi,\beta})$	$4\times 7+7^2+3\times 7^3+5\times 7^4+O(7^5)$	
$c_3(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+4\times 7^2+2\times 7^3+O(7^4)$	
$c_4(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+3\times 7^2+O(7^3)$	
$c_5(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+O(7^2)$	
$c_6(\mathcal{F}_{\chi,\beta})$	$0+O(7^1)$	
x_1	$\mathcal{A}\ +\ y\ +\ z$	where $y=\left(\mathcal{B}+\sqrt{\Delta}\right)^{1/3},\ z=\left(\mathcal{B}-\sqrt{\Delta}\right)^{1/3}$
x_2	$\mathcal{A}+\xi_3 y+\xi_3^2 z$	with $\mathcal{A}=0+O(7^2),\ \mathcal{B}=3\times 7+O(7^2)\mathbb{Q}quad$
x_3	$\mathcal{A}+\xi_3^2 y+\xi_3 z$	and $\Delta=2\times 7^2+O(7^3)\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad$

$\mathfrak{f} = 559 = \frac{(-7)^2+3\times 27^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+5\times 7^2+4\times 7^3+2\times 7^4+3\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+5\times 7^2+2\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+2\times 7^2+5\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+4\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$6\times 7+3\times 7^2+7^3+5\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 571 = \frac{(-31)^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+7^2+5\times 7^3+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+6\times 7^2+7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$4\times 7+6\times 7^2+7^3+5\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 577 = \frac{11^2+3\times 27^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7^2+2\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+4\times 7^2+4\times 7^3+2\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+5\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+2\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$3\times 7^2+6\times 7^3+4\times 7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 577 = \frac{11^2+3\times 27^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+4\times 7^3+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+6\times 7^2+7^3+5\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+7+4\times 7^2+4\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6\times 7+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times 7+7^2+O(7^3)$

$\mathfrak{f} = 601 = \frac{26^2+3\times 24^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+6\times 7^3+2\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7^2+4\times 7^3+7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+3\times 7^2+4\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+4\times 7+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$7+7^2+O(7^3)$

$\mathfrak{f} = 619 = \frac{17^2+3\times 27^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+7+7^2+6\times 7^3+7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+4\times 7^2+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$2\times 7+3\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 643 = \frac{(-40)^2+3\times 18^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+5\times 7^3+6\times 7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+7^3+4\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+7+6\times 7^2+2\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+7^2+O(7^3)$

$\mathfrak{f} = 657 = \frac{51^2+3\times 3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+6\times 7^2+7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+5\times 7^2+6\times 7^3+4\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7^2+3\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$2\times 7+2\times 7^2+5\times 7^3+7^4+7^5+O(7^6)$

$\mathfrak{f} = 661 = \frac{(-49)^2+3\times 9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+4\times 7^3+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+7+4\times 7^2+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+5\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times 7+2\times 7^2+5\times 7^3+2\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 661 = \frac{(-49)^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+7^3+2\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+5\times 7^2+3\times 7^3+6\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+2\times 7^2+3\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4\times 7+5\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2++O(7^1)$
x_1	$6\times 7+4\times 7^2+5\times 7^3+6\times 7^5+O(7^6)$

$\mathfrak{f} = 691 = \frac{8^2+3\times 30^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+6\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+2\times 7^2+4\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7^2+3\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+3\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$4\times 7^2+6\times 7^3+4\times 7^5+O(7^6)$

$\mathfrak{f}=709=\frac{53^2+3\times3^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times7+6\times7^2+7^3+4\times7^4+6\times7^5+6\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times7+4\times7^2+2\times7^3+5\times7^4+3\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+2\times7+5\times7^2+4\times7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+5\times7^2+5\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+6\times7+2\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5\times7^2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=5\times7+4\times7^2+O(7^3)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=3\times7+1\times7^2+O(7^3)$

$\mathfrak{f}=711=\frac{(-39)^2+3\times21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7+5\times7^2+7^3+2\times7^4+7^5+4\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+4\times7+5\times7^2+3\times7^3+6\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+3\times7+5\times7^2+3\times7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times7+4\times7^2+2\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+2\times7+2\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+2\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$7+7^2+6\times7^3+3\times7^4+O(7^6)$

$\mathfrak{f}=721=\frac{47^2+3\times15^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=103$

$\mathfrak{f}=721=\frac{(-34)^2+3\times24^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f}=103$

$\mathfrak{f}=733=\frac{50^2+3\times12^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7^2+6\times7^4+5\times7^5+2\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times7^3+3\times7^4+5\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3\times7+6\times7^2+7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times7+7^2+4\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+6\times7+4\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+2\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times7^2+5\times7^3+5\times7^4+3\times7^5+O(7^6)$

$\mathfrak{f}=769=\frac{(-49)^2+3\times15^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7^3+2\times7^5+4\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times7+7^2+5\times7^3+5\times7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times7+4\times7^2+4\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5\times7+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+4\times7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+2\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$7^2+O(7^3)$

$\mathfrak{f}=819=\frac{(-57)^2+3\times3^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=117$

$\mathfrak{f} = 819 = \frac{51^2+3\times 15^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f} = 117$

$\mathfrak{f} = 853 = \frac{35^2+3\times 27^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+4\times 7^3+3\times 7^4+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+7^3+6\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+4\times 7^2+5\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+2\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=1\times 7+1\times 7^2+O(7^3)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=4\times 7+2\times 7^2+O(7^3)$

$\mathfrak{f} = 871 = \frac{(-28)^2+3\times 30^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+7^3+7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+5\times 7^2+4\times 7^3+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+5\times 7^2+6\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$3\times 7+5\times 7^2+7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 873 = \frac{42^2+3\times 24^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+3\times 7^3+6\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7+6\times 7^2+5\times 7^3+7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4\times 7+2\times 7^2+3\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+5\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=0+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=7+O(7^2)$

$\mathfrak{f} = 877 = \frac{59^2+3\times 3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+6\times 7^3+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+6\times 7^2+5\times 7^3+5\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+6\times 7^2+5\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+7+6\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$6\times 7+5\times 7^2+4\times 7^3+6\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 883 = \frac{47^2+3\times21^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7^2+3\times7^3+5\times7^4+4\times7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times7+3\times7^2+4\times7^3+2\times7^4+6\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times7+3\times7^2+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+4\times7+4\times7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+5\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$5\times7+6\times7^2+O(7^3)$

$\mathfrak{f} = 937 = \frac{(-61)^2+3\times3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7+5\times7^2+4\times7^3+2\times7^4+3\times7^5+5\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+7+4\times7^2+5\times7^3+6\times7^4+5\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+7+6\times7^2+5\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+5\times7+3\times7^2+3\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times7+5\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$4\times7+3\times7^2+3\times7^3+6\times7^4+2\times7^5+O(7^6)$

$\mathfrak{f} = 949 = \frac{23^2+3\times33^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+7^2+2\times7^3+4\times7^4+3\times7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times7+7^2+3\times7^3+5\times7^4+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+5\times7+3\times7^2+6\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4\times7+6\times7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times7+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+5\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=4\times7+6\times7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=6\times7+1\times7^2+O(7^3)$

$\mathfrak{f} = 949 = \frac{(-58)^2+3\times12^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times7+3\times7^2+6\times7^4+5\times7^5+2\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times7+3\times7^3+5\times7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+6\times7+5\times7^2+2\times7^3+4\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+4\times7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+7+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$5\times7+3\times7^2+7^3+3\times7^4+3\times7^5+O(7^6)$

$\mathfrak{f} = 981 = \frac{51^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+7^3+4\times 7^4+6\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+2\times 7^2+7^3+7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+5\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$3\times 7+3\times 7^2+3\times 7^3+7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 991 = \frac{(-61)^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+4\times 7^2+7^3+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+7^2+4\times 7^3+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+2\times 7^2+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+6\times 7^2+4\times 7^3+7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1027 = \frac{56^2+3\times 18^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+7^2+3\times 7^3+4\times 7^4+5\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+2\times 7^2+3\times 7^3+2\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+6\times 7^2+7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7+5\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+4\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$2\times 7+7^2+2\times 7^3+7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 1057 = \frac{(-31)^2+3\times 33^2}{4},\ p=5,\ \beta=3$	see $\mathfrak{f} = 151$

$\mathfrak{f} = 1057 = \frac{50^2+3\times 24^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f} = 151$

$\mathfrak{f} = 1117 = \frac{65^2+3\times 9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+5\times 7^2+3\times 7^3+2\times 7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+4\times 7^2+3\times 7^3+4\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+4\times 7^2+7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+7+5\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$5\times 7+2\times 7^2+5\times 7^3+2\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 1117 = \frac{65^2+3\times 9^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+5\times 7^5+3\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^2+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+4\times 7^2+5\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+4\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 1123 = \frac{35^2+3\times 33^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+4\times 7^2+3\times 7^3+6\times 7^4+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+5\times 7^2+3\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+5\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3\times 7+2\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$7+O(7^1)$
x_1	$6\times 7+4\times 7^2+4\times 7^3+3\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 1129 = \frac{(-67)^2+3\times 3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+3\times 7^3+2\times 7^4+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+3\times 7^3+5\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+6\times 7^2+2\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+2\times 7^2+O(7^3)$

$\mathfrak{f} = 1143 = \frac{(-57)^2+3\times 21^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+4\times 7^3+6\times 7^4+5\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+6\times 7^2+3\times 7^3+7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6\times 7+3\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$2\times 7+O(7^3)$

$\mathfrak{f} = 1147 = \frac{5^2+3\times 39^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^3+7^4+4\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+6\times 7^3+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+7+5\times 7^2+2\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+4\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$5\times 7+5\times 7^2+5\times 7^3+7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1147 = \frac{(-49)^2+3\times 27^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+5\times 7^3+2\times 7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+2\times 7^2+2\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+7+3\times 7^2+6\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+5\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$6\times 7+5\times 7^2+5\times 7^3+2\times 7^5+O(7^6)$

$\mathfrak{f} = 1159 = \frac{(-37)^2+3\times 33^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+6\times 7^2+6\times 7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+7^2+5\times 7^3+5\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+3\times 7+4\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$2\times 7+6\times 7^2+7^3+2\times 7^5+O(7^6)$

$\mathfrak{f} = 1159 = \frac{(-37)^2+3\times 33^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+7^2+5\times 7^4+5\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+5\times 7^2+4\times 7^3+6\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+6\times 7^2+4\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7+5\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+7^2+5\times 7^3+7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 1213 = \frac{17^2+3\times 39^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+5\times 7^3+6\times 7^4+7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7+7^2+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+6\times 7^2+4\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+3\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times 7+5\times 7^2+O(7^3)$

$\mathfrak{f} = 1231 = \frac{(-19)^2+3\times 39^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+3\times 7+4\times 7^3+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4\times 7+7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+7+2\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$2\times 7+6\times 7^2+4\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1237 = \frac{41^2+3\times 33^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+6\times 7^3+4\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+3\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+3\times 7^2+7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+4\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=3\times 7+4\times 7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=4\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 1273 = \frac{(-58)^2+3\times 24^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+4\times 7^2+4\times 7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+3\times 7^2+2\times 7^3+5\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+6\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2\times 7+3\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+4\times 7^2+3\times 7^3+3\times 7^5+O(7^6)$

$\mathfrak{f} = 1279 = \frac{(-43)^2+3\times 33^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+2\times 7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+6\times 7^2+6\times 7^3+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+7+2\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4\times 7+7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times 7+5\times 7^2+4\times 7^3+3\times 7^5+O(7^6)$

$\mathfrak{f} = 1279 = \frac{(-43)^2+3\times 33^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^3+4\times 7^4+3\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+4\times 7^2+3\times 7^3+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+5\times 7^2+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$2\times 7^2+O(7^3)$

$\mathfrak{f} = 1297 = \frac{(-25)^2+3\times 39^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+6\times 7^3+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+2\times 7^2+6\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+3\times 7^2+2\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$6\times 7+5\times 7^2+7^3+5\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 1303 = \frac{(-55)^2+3\times 27^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+4\times 7^2+7^3+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+7^2+4\times 7^3+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+7^2+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+2\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$2\times 7+4\times 7^2+6\times 7^3+2\times 7^4+O(7^6)$

$\mathfrak{f}=1321=\frac{71^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+2\times 7^2+5\times 7^3+6\times 7^4+3\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+6\times 7^2+7^3+7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+7+5\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+4\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$4\times 7+6\times 7^2+2\times 7^5+O(7^6)$

$\mathfrak{f}=1339=\frac{8^2+3\times 42^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^2+6\times 7^3+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+4\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+2\times 7^2+5\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+5\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$6\times 7+2\times 7^2+4\times 7^3+2\times 7^5+O(7^6)$

$\mathfrak{f}=1339=\frac{8^2+3\times 42^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+4\times 7^2+5\times 7^3+5\times 7^4+5\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+4\times 7^2+2\times 7^3+2\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+3\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$4\times 7+4\times 7^2+6\times 7^3+6\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f}=1351=\frac{(-52)^2+3\times 30^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=193$

$\mathfrak{f}=1351=\frac{29^2+3\times 39^2}{4},\ p=5,\ \beta=3$	see $\mathfrak{f}=193$

$\mathfrak{f}=1351=\frac{(-52)^2+3\times 30^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f}=193$

$\mathfrak{f}=1351=\frac{29^2+3\times 39^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f}=193$

$\mathfrak{f}=1381=\frac{(-31)^2+3\times 39^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+6\times 7^3+3\times 7^4+2\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+6\times 7^3+5\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+4\times 7^2+2\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+4\times 7+6\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$3\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f}=1387=\frac{(-16)^2+3\times 42^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+7^2+4\times 7^3+7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+5\times 7^2+4\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+4\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+6\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$6\times 7+2\times 7^2+3\times 7^3+7^4+7^5+O(7^6)$

$\mathfrak{f}=1387=\frac{(-16)^2+3\times 42^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+3\times 7^3+3\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7+3\times 7^3+2\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+5\times 7^2+6\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$4\times 7+5\times 7^2+O(7^3)$

$\mathfrak{f}=1417=\frac{(-49)^2+3\times 33^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+3\times 7^3+7^4+6\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+3\times 7^2+4\times 7^3+4\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+4\times 7^2+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+3\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$2\times 7+7^2+6\times 7^3+2\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f}=1447=\frac{35^2+3\times 39^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+7^4+3\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7+3\times 7^2+7^3+7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+6\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=7+O(7^3)\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=3\times 7+5\times 7^2+O(7^3)$

$\mathfrak{f} = 1459 = \frac{56^2+3\times 30^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+4\times 7^2+4\times 7^3+3\times 7^4+5\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+7^3+4\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+7^2+2\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+5\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+3\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=5\times 7+4\times 7^2+O(7^3)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=4\times 7+5\times 7^2+O(7^3)$

$\mathfrak{f} = 1483 = \frac{(-37)^2+3\times 39^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^2+4\times 7^3+5\times 7^4+6\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+7+3\times 7^2+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+7^2+3\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5\times 7+4\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$5\times 7+4\times 7^2+2\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 1489 = \frac{77^2+3\times 3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+7^3+5\times 7^4+4\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+4\times 7^2+4\times 7^3+3\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+2\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$5\times 7^2+7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 1543 = \frac{77^2+3\times 9^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+6\times 7^3+4\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+7+7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+3\times 7+4\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$7+4\times 7^2+6\times 7^3+7^4+5\times 7^5+O(7^6)$

$\mathfrak{f}=1543=\frac{77^2+3\times 9^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+6\times 7^3+5\times 7^4+7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7+7^3+4\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+3\times 7^2+5\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+6\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$2\times 7+O(7^3)$

$\mathfrak{f}=1561=\frac{(-13)^2+3\times 45^2}{4},\ p=7,\ \beta=3$	see $\mathfrak{f}=223$

$\mathfrak{f}=1561=\frac{41^2+3\times 39^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f}=223$

$\mathfrak{f}=1567=\frac{(-79)^2+3\times 3^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+3\times 7^3+3\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+5\times 7^2+5\times 7^3+6\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+2\times 7^2+6\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+7+5\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+7^2+2\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f}=1567=\frac{(-79)^2+3\times 3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+4\times 7^3+2\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times 7+6\times 7^2+O(7^3)$

$\mathfrak{f}=1579=\frac{32^2+3\times 42^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+4\times 7^3+5\times 7^4+2\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+6\times 7^2+2\times 7^3+3\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+6\times 7^2+7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+3\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$2\times 7+6\times 7^2+4\times 7^3+7^4+3\times 7^5+O(7^6)$

$\mathfrak{f}=1603=\frac{(-43)^2+3\times 39^2}{4},\ p=7,\ \beta=3$	see $\mathfrak{f}=229$

$\mathfrak{f} = 1603 = \frac{65^2+3\times 27^2}{4},\ p = 7,\ \beta = 5$	see $\mathfrak{f} = 229$

$\mathfrak{f} = 1629 = \frac{(-57)^2+3\times 33^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 6\times 7^2 + 5\times 7^3 + 2\times 7^4 + 5\times 7^5 + 3\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4 + 6\times 7 + 2\times 7^2 + 3\times 7^3 + 4\times 7^4 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7 + 2\times 7^3 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3 + 6\times 7 + 4\times 7^2 + 2\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6 + 2\times 7 + 4\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5 + 4\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$4\times 7 + 4\times 7^2 + 7^4 + O(7^6)$

$\mathfrak{f} = 1651 = \frac{23^2+3\times 45^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2 + 5\times 7^3 + 7^4 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 2\times 7^2 + 7^3 + 7^4 + 3\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1 + 3\times 7 + 4\times 7^2 + 3\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 2\times 7 + 4\times 7^2 + 4\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4 + 5\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6 + O(7^1)$
x_1	$3\times 7 + 4\times 7^2 + O(7^3)$

$\mathfrak{f} = 1657 = \frac{(-70)^2+3\times 24^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7 + 4\times 7^4 + 3\times 7^5 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 5\times 7^2 + 5\times 7^3 + 2\times 7^4 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4 + 7^2 + 6\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1 + 3\times 7 + 4\times 7^2 + 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1 + 2\times 7 + 3\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2 + 6\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4 + O(7^1)$
x_1	$\mathcal{A} + \sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A} = 2\times 7 + 2\times 7^2 + O(7^3)$
x_2	$\mathcal{A} - \sqrt{\Delta}\mathbb{Q}uad$ and $\Delta = 2\times 7 + 4\times 7^2 + O(7^3)$

$\mathfrak{f} = 1663 = \frac{(-73)^2+3\times 21^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^4 + 7^5 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 2\times 7^2 + 7^4 + 5\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5 + 2\times 7 + 4\times 7^3 + 4\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 4\times 7 + 5\times 7^2 + 3\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2 + 3\times 7 + 5\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5 + O(7^1)$
x_1	$O(7^3)$

$\mathfrak{f} = 1687 = \frac{59^2+3\times 33^2}{4},\ p = 5,\ \beta = 3$	see $\mathfrak{f} = 241$

$\mathfrak{f} = 1687 = \frac{(-76)^2+3\times 18^2}{4},\ p = 7,\ \beta = 5$	see $\mathfrak{f} = 241$

$\mathfrak{f} = 1693 = \frac{47^2+3\times 39^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+2\times 7^3+7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+3\times 7^2+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+5\times 7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$4\times 7+6\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1737 = \frac{(-75)^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+5\times 7^2+7^3+2\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+3\times 7^2+6\times 7^3+2\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7^2+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$3\times 7+3\times 7^3+6\times 7^4+O(7^6)$

$\mathfrak{f} = 1741 = \frac{(-49)^2+3\times 39^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+2\times 7^3+6\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+4\times 7^2+4\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+6\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+O(7^3)$

$\mathfrak{f} = 1759 = \frac{(-31)^2+3\times 45^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7^2+6\times 7^3+5\times 7^4+7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+3\times 7^2+2\times 7^3+2\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+5\times 7^2+6\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+2\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$5\times 7+2\times 7^2+O(7^3)$

$\mathfrak{f} = 1777 = \frac{14^2+3\times 48^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2 + 7^3 + 5\times 7^4 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times 7 + 6\times 7^3 + 6\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2 + 7 + 2\times 7^3 + 5\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 3\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2 + 6\times 7 + 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1 + 6\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$7 + O(7^1)$
x_1	$6\times 7 + 3\times 7^2 + O(7^3)$

$\mathfrak{f} = 1783 = \frac{83^2+3\times 9^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 2\times 7^2 + 2\times 7^3 + 3\times 7^4 + 6\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5 + 5\times 7 + 3\times 7^2 + 7^3 + 3\times 7^4 + 3\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1 + 4\times 7 + 4\times 7^2 + 7^3 + 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 2\times 7 + 7^2 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4 + O(7^1)$
x_1	$5\times 7 + 2\times 7^3 + 4\times 7^4 + 2\times 7^5 + O(7^6)$

$\mathfrak{f} = 1783 = \frac{83^2+3\times 9^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 6\times 7^2 + 7^4 + 4\times 7^5 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3 + 4\times 7^2 + 7^3 + 5\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2 + 4\times 7 + 6\times 7^2 + 6\times 7^3 + 5\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5 + 3\times 7 + 7^2 + 6\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4 + 3\times 7 + 5\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1 + 2\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2 + O(7^1)$
x_1	$3\times 7 + 3\times 7^2 + 2\times 7^4 + 4\times 7^5 + O(7^6)$

$\mathfrak{f} = 1801 = \frac{74^2+3\times 24^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 7^2 + 2\times 7^3 + 4\times 7^4 + 3\times 7^5 + 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4 + 6\times 7^2 + 3\times 7^3 + 5\times 7^4 + 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1 + 4\times 7 + 6\times 7^2 + 2\times 7^3 + 3\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3 + 7 + 6\times 7^2 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2\times 7 + 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4 + 2\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1 + O(7^1)$
x_1	$7 + 7^2 + 6\times 7^3 + 5\times 7^5 + O(7^6)$

$\mathfrak{f} = 1831 = \frac{68^2+3\times 30^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+3\times 7^2+4\times 7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+7^2+2\times 7^3+2\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+5\times 7^2+2\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$6\times 7+7^2+4\times 7^3+2\times 7^4+7^5+O(7^6)$

$\mathfrak{f} = 1879 = \frac{(-73)^2+3\times 27^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^3+6\times 7^4+4\times 7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7+4\times 7^2+2\times 7^3+5\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+6\times 7^2+7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$5\times 7^2+O(7^3)$

$\mathfrak{f} = 1891 = \frac{83^2+3\times 15^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+3\times 7^3+3\times 7^4+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+7^3+2\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+7+3\times 7^2+4\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$4\times 7+3\times 7^2+3\times 7^3+6\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1899 = \frac{(-48)^2+3\times 42^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+6\times 7^2+5\times 7^3+7^4+4\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+6\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+2\times 7^2+5\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7+6\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$3\times 7+2\times 7^2+2\times 7^3+5\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 1957 = \frac{86^2+3\times12^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+4\times7^3+5\times7^4+2\times7^5+5\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+6\times7^3+2\times7^4+6\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+7+3\times7^2+2\times7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+7^2+2\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+2\times7+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$2\times7+O(7^3)$

$\mathfrak{f} = 1993 = \frac{(-13)^2+3\times51^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7+3\times7^3+5\times7^4+6\times7^5+3\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+7+5\times7^4+5\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+5\times7+5\times7^2+5\times7^3+3\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$4\times7+7^2+6\times7^4+2\times7^5+O(7^6)$

$\mathfrak{f} = 1993 = \frac{(-13)^2+3\times51^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+7^3+5\times7^4+4\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times7+4\times7^2+5\times7^3+2\times7^4+5\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times7+4\times7^2+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+2\times7^2+4\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+7+4\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$6\times7+5\times7^2+O(7^3)$

$\mathfrak{f} = 2007 = \frac{15^2+3\times51^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7+6\times7^3+5\times7^4+2\times7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+4\times7+4\times7^2+4\times7^3+2\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+6\times7+2\times7^2+5\times7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+4\times7+2\times7^2+2\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$5\times7+6\times7^2+3\times7^3+5\times7^4+3\times7^5+O(7^6)$

$\mathfrak{f} = 2053 = \frac{83^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+5\times 7^3+3\times 7^4+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+6\times 7^2+7^3+4\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+5\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+7+3\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$6\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 2061 = \frac{87^2+3\times 15^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+3\times 7^2+3\times 7^3+2\times 7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+6\times 7^2+2\times 7^3+5\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+6\times 7^2+3\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$6\times 7+7^2+4\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 2061 = \frac{87^2+3\times 15^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+5\times 7^3+6\times 7^4+6\times 7^5+3\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+4\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+7+5\times 7^2+5\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$2\times 7+7^2+O(7^3)$

$\mathfrak{f} = 2077 = \frac{(-91)^2+3\times 3^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+5\times 7^2+7^3+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+2\times 7^3+4\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+2\times 7^2+5\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+2\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$7+O(7^1)$
x_1	$6\times 7+5\times 7^2+5\times 7^3+5\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 2119 = \frac{(-76)^2+3\times 30^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+6\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+4\times 7^2+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+2\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+6\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=6\times 7+5\times 7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=7+3\times 1\times 7^2+O(7^3)$

$\mathfrak{f} = 2119 = \frac{(-76)^2+3\times 30^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+2\times 7^3+2\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+3\times 7^2+7^3+5\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+7+4\times 7^2+6\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+5\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$4\times 7+7^2+O(7^3)$

$\mathfrak{f} = 2119 = \frac{(-49)^2+3\times 45^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+5\times 7^4+5\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+4\times 7^3+6\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+5\times 7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+4\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+7^2+O(7^3)$

$\mathfrak{f} = 2131 = \frac{(-91)^2+3\times 9^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+3\times 7^3+2\times 7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+4\times 7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=3\times 7+2\times 7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=2\times 7+4\times 7^2+O(7^3)$

$\mathfrak{f} = 2169 = \frac{42^2+3\times 48^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+2\times 7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+4\times 7^2+6\times 7^3+5\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+5\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$4\times 7+3\times 7^2+5\times 7^3+3\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 2169 = \frac{(-93)^2+3\times 3^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+4\times 7^4+3\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+4\times 7^2+2\times 7^3+5\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+3\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$7^2+7^4+O(7^6)$

$\mathfrak{f} = 2169 = \frac{42^2+3\times 48^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+5\times 7^3+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+7+5\times 7^2+7^3+6\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+7+4\times 7^2+5\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2\times 7+3\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$7+5\times 7^2+7^3+5\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 2191 = \frac{(-4)^2+3\times 54^2}{4},\ p=5,\ \beta=1$	see $\mathfrak{f} = 313$

$\mathfrak{f} = 2191 = \frac{(-31)^2+3\times 51^2}{4},\ p=5,\ \beta=3$	see $\mathfrak{f} = 313$

$\mathfrak{f} = 2191 = \frac{(-4)^2+3\times 54^2}{4},\ p=5,\ \beta=3$	see $\mathfrak{f} = 313$

$\mathfrak{f} = 2191 = \frac{(-31)^2+3\times 51^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f} = 313$

$\mathfrak{f} = 2203 = \frac{8^2+3\times 54^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+2\times 7^3+3\times 7^4+7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+6\times 7^3+2\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=1\times 7+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=2\times 7+O(7^2)$

$\mathfrak{f}=2223=\frac{33^2+3\times 51^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+6\times 7^3+2\times 7^4+6\times 7^5+6\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+6\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5\times 7+4\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+3\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$0+O(7^1)$
x_1	$\mathcal{A}\ +\ y\ +\ z$ where $y=\left(\mathcal{B}+\sqrt{\Delta}\right)^{1/3},\ z=\left(\mathcal{B}-\sqrt{\Delta}\right)^{1/3}$
x_2	$\mathcal{A}+\xi_3 y+\xi_3^2 z$ with $\mathcal{A}=0+O(7^2),\ \mathcal{B}=7+O(7^2)\mathbb{Q}quad\mathbb{Q}quad$
x_3	$\mathcal{A}+\xi_3^2 y+\xi_3 z$ and $\Delta=7^2+O(7^3)\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad$

$\mathfrak{f}=2223=\frac{33^2+3\times 51^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7^2+6\times 7^3+3\times 7^4+4\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+6\times 7^3+2\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2\times 7+4\times 7^2+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+7+2\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=0+O(7^2)\mathbb{Q}quad$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=2\times 7+O(7^2)$

$\mathfrak{f}=2239=\frac{(-91)^2+3\times 15^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+6\times 7^3+4\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1+3\times 7+5\times 7^2+7^3+3\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+5\times 7^2+6\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+2\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$2\times 7+7^2+7^3+7^4+7^5+O(7^6)$

$\mathfrak{f}=2263=\frac{95^2+3\times 3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^4+5\times 7^5+5\times 7^6+4\times 7^7+5\times 7^8+3\times 7^9+O(7^{10})$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+5\times 7^3+7^4+5\times 7^5+3\times 7^6+5\times 7^7+4\times 7^8+O(7^9)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3\times 7+5\times 7^2+6\times 7^3+6\times 7^4+6\times 7^5+3\times 7^7+O(7^8)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+2\times 7^2+4\times 7^3+4\times 7^5+4\times 7^6+O(7^7)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+4\times 7+4\times 7^3+4\times 7^4+4\times 7^5+O(7^6)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+2\times 7^2+7^3+6\times 7^4+O(7^5)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6\times 7+2\times 7^3+O(7^4)$
$c_7(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+O(7^3)$
$c_8(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+O(7^2)$
$c_9(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=7+7^2+O(7^3)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=7^2+O(7^3)\mathbb{Q}quad$

$\mathfrak{f} = 2269 = \frac{83^2+3\times27^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times7^2+2\times7^3+5\times7^4+3\times7^5+5\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+4\times7^2+3\times7^3+3\times7^4+3\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+3\times7+5\times7^2+5\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times7+3\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times7+4\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times7^2+7^3+4\times7^4+4\times7^5+O(7^6)$

$\mathfrak{f} = 2287 = \frac{20^2+3\times54^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times7^2+2\times7^3+7^4+3\times7^5+5\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times7+7^2+3\times7^3+3\times7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+6\times7+5\times7^2+7^3+2\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times7+4\times7^2+6\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+2\times7+3\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+6\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$2\times7+4\times7^2+O(7^3)$

$\mathfrak{f} = 2311 = \frac{89^2+3\times21^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times7^2+7^3+6\times7^4+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+2\times7+4\times7^2+2\times7^3+3\times7^4+3\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+2\times7+7^2+6\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2\times7+2\times7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+3\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$7^2+2\times7^3+O(7^6)$

$\mathfrak{f} = 2331 = \frac{(-39)^2+3\times51^2}{4},\ p = 7,\ \beta = 1$	see $\mathfrak{f} = 333$

$\mathfrak{f} = 2331 = \frac{69^2+3\times39^2}{4},\ p = 7,\ \beta = 3$	see $\mathfrak{f} = 333$

$\mathfrak{f} = 2347 = \frac{(-64)^2+3\times42^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times7^2+7^3+2\times7^4+4\times7^5+2\times7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times7+7^2+6\times7^3+4\times7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$7+4\times7^2+6\times7^3+6\times7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+5\times7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+5\times7+6\times7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+4\times7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$7+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=1\times7+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=2\times7+O(7^2)$

$\mathfrak{f} = 2359 = \frac{92^2+3\times18^2}{4},\ p = 7,\ \beta = 3$	see $\mathfrak{f} = 337$

$\mathfrak{f} = 2359 = \frac{(-97)^2+3\times 3^2}{4},\ p=7,\ \beta=5$	see $\mathfrak{f} = 337$

$\mathfrak{f} = 2377 = \frac{(-79)^2+3\times 33^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+4\times 7^2+4\times 7^3+2\times 7^4+3\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+7^2+4\times 7^3+2\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+3\times 7^2+4\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+4\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+2\times 7^3+3\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 2413 = \frac{(-43)^2+3\times 51^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+6\times 7^2+3\times 7^3+2\times 7^4+7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+6\times 7^2+7^3+2\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+7^2+6\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$3\times 7+4\times 7^2+2\times 7^3+3\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 2439 = \frac{(-3)^2+3\times 57^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+5\times 7^3+6\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+3\times 7^2+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+6\times 7^2+3\times 7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+6\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$3\times 7+4\times 7^3+2\times 7^4+6\times 7^5+O(7^6)$

$\mathfrak{f} = 2439 = \frac{(-3)^2+3\times 57^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+6\times 7^3+7^4+5\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+4\times 7^3+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+5\times 7^2+7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+4\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$2\times 7+5\times 7^2+4\times 7^3+4\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 2439 = \frac{(-84)^2+3\times 30^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+6\times 7^3+4\times 7^4+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+2\times 7^3+4\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+3\times 7+7^2+6\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+6\times 7+5\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 2449 = \frac{(-61)^2+3\times 45^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+4\times 7^3+2\times 7^4+3\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+6\times 7^2+5\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+3\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$5\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 2479 = \frac{(-13)^2+3\times 57^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7^2+5\times 7^3+4\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+2\times 7^2+6\times 7^3+6\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+3\times 7^2+7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+3\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$5\times 7+4\times 7^2+O(7^3)$

$\mathfrak{f} = 2493 = \frac{(-93)^2+3\times 21^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+5\times 7^2+7^3+6\times 7^4+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+2\times 7^2+4\times 7^3+6\times 7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+7^2+2\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+7+4\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times 7+6\times 7^2+5\times 7^3+2\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 2493 = \frac{15^2+3\times 57^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7+5\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+3\times 7+5\times 7^2+6\times 7^3+5\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+5\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$7+3\times 7^3+4\times 7^4+O(7^6)$

$\mathfrak{f} = 2493 = \frac{(-93)^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+7^2+4\times 7^3+3\times 7^4+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7+5\times 7^2+7^3+5\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+7+3\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1 x_2 x_3	$\mathcal{A}\ +\ y\ +\ z$ where $y=\left(\mathcal{B}+\sqrt{\Delta}\right)^{1/3},\ z=\left(\mathcal{B}-\sqrt{\Delta}\right)^{1/3}$ $\mathcal{A}+\xi_3 y+\xi_3^2 z$ with $\mathcal{A}=7+O(7^2),\ \mathcal{B}=5\times 7+O(7^2)\mathbb{Q}quad$ $\mathcal{A}+\xi_3^2 y+\xi_3 z$ and $\Delta=4\times 7^2+O(7^3)\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad\mathbb{Q}quad$

$\mathfrak{f} = 2503 = \frac{47^2+3\times 51^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+2\times 7^3+7^4+6\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6\times 7+4\times 7^2+6\times 7^3+4\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+2\times 7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+2\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$2\times 7+7^2+O(7^3)$

$\mathfrak{f} = 2547 = \frac{(-21)^2+3\times 57^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+2\times 7^2+7^3+3\times 7^4+6\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+5\times 7^2+5\times 7^3+6\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+5\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6\times 7+6\times 7^2+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$2\times 7+3\times 7^2+3\times 7^3+3\times 7^4+O(7^6)$

$\mathfrak{f} = 2587 = \frac{95^2+3\times 21^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+6\times 7^3+6\times 7^4+2\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4\times 7+7^3+7^4+6\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3\times 7+3\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+6\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+5\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=4\times 7+O(7^2)$
x_2	$\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=4\times 7+O(7^2)$

$\mathfrak{f} = 2623 = \frac{77^2+3\times 39^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+7+4\times 7^2+5\times 7^3+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+5\times 7^2+7^3+4\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+5\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$7+5\times 7^2+5\times 7^3+3\times 7^5+O(7^6)$

$\mathfrak{f} = 2677 = \frac{(-31)^2+3\times 57^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+4\times 7^3+7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7^2+3\times 7^3+7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2+2\times 7^2+6\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$7+2\times 7^2+O(7^3)$

$\mathfrak{f} = 2683 = \frac{(-97)^2+3\times 21^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^3+6\times 7^4+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+3\times 7^2+5\times 7^3+2\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5\times 7+5\times 7^2+3\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+4\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$7+2\times 7^2+4\times 7^3+6\times 7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 2701 = \frac{(-79)^2+3\times 39^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 3\times 7^2 + 2\times 7^4 + 2\times 7^5 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5 + 5\times 7 + 2\times 7^2 + 6\times 7^3 + 3\times 7^4 + 2\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1 + 2\times 7 + 6\times 7^3 + 2\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 2\times 7^2 + 4\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3 + 6\times 7 + 4\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$5\times 7 + 4\times 7^2 + 6\times 7^3 + 4\times 7^5 + O(7^6)$

$\mathfrak{f} = 2707 = \frac{(-55)^2+3\times 51^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 4\times 7^2 + 2\times 7^3 + 5\times 7^4 + 6\times 7^5 + 3\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4 + 3\times 7 + 4\times 7^2 + 4\times 7^4 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6 + 4\times 7 + 7^2 + 3\times 7^3 + 2\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1 + 5\times 7 + 7^2 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2 + 5\times 7 + 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5 + O(7^1)$
x_1	$7 + 2\times 7^3 + 6\times 7^4 + 7^5 + O(7^6)$

$\mathfrak{f} = 2713 = \frac{(-103)^2+3\times 9^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2 + 5\times 7^3 + 4\times 7^4 + 5\times 7^5 + 2\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5 + 6\times 7 + 7^2 + 6\times 7^3 + 5\times 7^4 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6 + 2\times 7 + 6\times 7^2 + 5\times 7^3 + 2\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6\times 7^2 + 6\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6 + 6\times 7 + 2\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4 + 3\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4 + O(7^1)$
x_1	$7^2 + 6\times 7^3 + 2\times 7^4 + O(7^6)$

$\mathfrak{f} = 2731 = \frac{104^2+3\times 6^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7 + 6\times 7^2 + 2\times 7^3 + 3\times 7^4 + 2\times 7^5 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5 + 6\times 7 + 3\times 7^2 + 3\times 7^3 + 5\times 7^4 + 2\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5 + 5\times 7^2 + 3\times 7^3 + 6\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 3\times 7 + 5\times 7^2 + 6\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$1 + 7 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5 + 4\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5 + O(7^1)$
x_1	$7 + 2\times 7^2 + 4\times 7^3 + O(7^6)$

$\mathfrak{f} = 2763 = \frac{(-57)^2+3\times 51^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+7^3+6\times 7^4+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+5\times 7^2+6\times 7^3+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+7^2+3\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+7+5\times 7^2+6\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+3\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$5\times 7^2+2\times 7^3+2\times 7^4+7^5+O(7^6)$

$\mathfrak{f} = 2803 = \frac{95^2+3\times 27^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+4\times 7^2+2\times 7^3+3\times 7^4+2\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+7^3+4\times 7^4+5\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+7^2+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+5\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$5\times 7+5\times 7^2+6\times 7^3+4\times 7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 2817 = \frac{(-39)^2+3\times 57^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7+5\times 7^2+6\times 7^3+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+3\times 7^2+6\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+6\times 7^2+4\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+3\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+5\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$7+5\times 7^2+3\times 7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 2821 = \frac{(-22)^2+3\times 60^2}{4},\ p=7,\ \beta=1$	see $\mathfrak{f}=403$

$\mathfrak{f} = 2821 = \frac{59^2+3\times 51^2}{4},\ p=7,\ \beta=3$	see $\mathfrak{f}=403$

$\mathfrak{f} = 2833 = \frac{98^2+3\times 24^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+6\times 7^2+2\times 7^4+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+7^2+6\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+7+6\times 7^2+4\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4\times 7+4\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+6\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$6\times 7+3\times 7^2+7^3+7^4+4\times 7^5+O(7^6)$

$\mathfrak{f} = 2869 = \frac{107^2+3\times 3^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+3\times 7^3+3\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+2\times 7^2+7^3+3\times 7^4+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+2\times 7+7^2+2\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3\times 7+2\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1	$5\times 7+2\times 7^2+O(7^3)$

$\mathfrak{f} = 2899 = \frac{(-43)^2+3\times 57^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+3\times 7^2+2\times 7^3+4\times 7^4+3\times 7^5+7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+4\times 7+3\times 7^2+4\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+5\times 7^2+7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+6\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+3\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$4+O(7^1)$
x_1	$4\times 7+3\times 7^2+6\times 7^3+4\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 2983 = \frac{(-103)^2+3\times 21^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2+2\times 7^4+3\times 7^5+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+6\times 7^2+7^3+3\times 7^4+4\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+2\times 7^2+5\times 7^3+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+5\times 7^2+7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+5\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times 7+5\times 7^2+O(7^3)$

$\mathfrak{f} = 3001 = \frac{77^2+3\times 45^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7^2+6\times 7^3+7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+4\times 7^3+3\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+7+4\times 7^3+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+3\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$6+O(7^1)$
x_1	$2\times 7+6\times 7^2+O(7^3)$

$\mathfrak{f} = 3007 = \frac{65^2+3\times 51^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$7+5\times 7^3+2\times 7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+3\times 7^2+2\times 7^3+3\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6+2\times 7+4\times 7^2+6\times 7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$3+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=1\times 7+7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=7+5\times 7^2+O(7^3)$

$\mathfrak{f} = 3033 = \frac{(-102)^2+3\times 24^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7+7^2+7^3+5\times 7^4+4\times 7^5+5\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$7+6\times 7^2+3\times 7^3+5\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+6\times 7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+2\times 7+2\times 7^2+2\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$2+4\times 7+4\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1 x_2	$\mathcal{A}+\sqrt{\Delta}\mathbb{Q}uad$ where $\mathcal{A}=0\times 7+2\times 7^2+O(7^3)$ $\mathcal{A}-\sqrt{\Delta}\mathbb{Q}uad$ and $\Delta=3\times 7+3\times 7^2+O(7^3)$

$\mathfrak{f} = 3033 = \frac{(-102)^2+3\times 24^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7^2+6\times 7^3+5\times 7^4+6\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2\times 7+7^2+6\times 7^3+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+5\times 7+2\times 7^2+7^3+5\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+2\times 7+4\times 7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+3\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1+O(7^1)$
x_1	$4\times 7+7^2+O(7^3)$

$\mathfrak{f} = 3037 = \frac{(-49)^2+3\times 57^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+6\times 7^2+7^3+6\times 7^4+4\times 7^5+2\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+5\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$5+4\times 7+5\times 7^2+6\times 7^3+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+6\times 7+6\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4+7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$3\times 7+4\times 7^2+7^3+7^4+5\times 7^5+O(7^6)$

$\mathfrak{f} = 3049 = \frac{17^2+3\times 63^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+6\times 7^2+5\times 7^3+7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+3\times 7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+4\times 7^2+7^3+3\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3+5\times 7+5\times 7^2+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+4\times 7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+4\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$6\times 7+4\times 7^2+2\times 7^3+5\times 7^4+O(7^6)$

$\mathfrak{f} = 3067 = \frac{(-19)^2+3\times 63^2}{4},\ p = 7,\ \beta = 5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7^2+4\times 7^4+6\times 7^5+4\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^3+7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3+6\times 7+2\times 7^2+5\times 7^3+6\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$5+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+7+7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$1+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$4\times 7+7^2+O(7^3)$

$\mathfrak{f} = 3097 = \frac{101^2+3\times 27^2}{4},\ p = 7,\ \beta = 1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$5\times 7+2\times 7^3+4\times 7^4+7^5+3\times 7^6+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$4+2\times 7+5\times 7^2+7^3+2\times 7^4+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$4+6\times 7+2\times 7^2+2\times 7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2+5\times 7+7^2+5\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6+7+2\times 7^2+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5+O(7^1)$
x_1	$4\times 7+4\times 7^2+7^3+5\times 7^4+3\times 7^5+O(7^6)$

$\mathfrak{f} = 3139 = \frac{53^2+3\times 57^2}{4},\ p = 7,\ \beta = 3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$4\times 7+4\times 7^2+2\times 7^3+7^4+O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$6+7+6\times 7^2+6\times 7^3+2\times 7^4+7^5+O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$1+4\times 7+2\times 7^2+7^4+O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6+3\times 7+4\times 7^2+4\times 7^3+O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3+3\times 7+O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$6+6\times 7+O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$2+O(7^1)$
x_1	$4\times 7+2\times 7^3+6\times 7^4+2\times 7^5+O(7^6)$

$\mathfrak{f} = 3139 = \frac{53^2+3\times 57^2}{4},\ p=7,\ \beta=5$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2 + 7^3 + 2\times 7^4 + 5\times 7^5 + 5\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$5\times 7 + 6\times 7^2 + 6\times 7^3 + 6\times 7^4 + 5\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$6 + 4\times 7 + 7^2 + 6\times 7^3 + 2\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$3\times 7 + 4\times 7^2 + 3\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3 + 5\times 7 + 3\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$2 + 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$1 + O(7^1)$
x_1	$6\times 7 + 2\times 7^2 + O(7^3)$

$\mathfrak{f} = 3141 = \frac{69^2+3\times 51^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7 + 2\times 7^3 + 2\times 7^4 + 4\times 7^5 + 3\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$3 + 2\times 7^2 + 3\times 7^3 + 3\times 7^4 + 2\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3 + 4\times 7^2 + 7^3 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$2 + 5\times 7 + 6\times 7^2 + 2\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$6\times 7 + 6\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4 + 6\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$O(7^1)$
x_1	$5\times 7 + 2\times 7^2 + 3\times 7^3 + 2\times 7^5 + O(7^6)$

$\mathfrak{f} = 3141 = \frac{42^2+3\times 60^2}{4},\ p=7,\ \beta=1$	
$c_0(\mathcal{F}_{\chi,\beta})$	$2\times 7 + 6\times 7^2 + 5\times 7^3 + 7^5 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$1 + 5\times 7 + 4\times 7^2 + 4\times 7^3 + 2\times 7^4 + 3\times 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$3 + 2\times 7 + 5\times 7^2 + 3\times 7^3 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$6 + 2\times 7 + 2\times 7^2 + 5\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$4 + 5\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$5 + 3\times 7 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$5 + O(7^1)$
x_1	$5\times 7 + 5\times 7^2 + 6\times 7^3 + 2\times 7^4 + 5\times 7^5 + O(7^6)$

$\mathfrak{f} = 3141 = \frac{42^2+3\times 60^2}{4},\ p=7,\ \beta=3$	
$c_0(\mathcal{F}_{\chi,\beta})$	$6\times 7^2 + 2\times 7^3 + 3\times 7^4 + 2\times 7^5 + 4\times 7^6 + O(7^7)$
$c_1(\mathcal{F}_{\chi,\beta})$	$2 + 5\times 7 + 5\times 7^2 + 4\times 7^3 + 4\times 7^4 + 7^5 + O(7^6)$
$c_2(\mathcal{F}_{\chi,\beta})$	$2\times 7 + 7^2 + 7^3 + 3\times 7^4 + O(7^5)$
$c_3(\mathcal{F}_{\chi,\beta})$	$4 + 5\times 7 + 3\times 7^2 + 6\times 7^3 + O(7^4)$
$c_4(\mathcal{F}_{\chi,\beta})$	$3 + 6\times 7 + 2\times 7^2 + O(7^3)$
$c_5(\mathcal{F}_{\chi,\beta})$	$4 + O(7^2)$
$c_6(\mathcal{F}_{\chi,\beta})$	$7 + O(7^1)$
x_1	$4\times 7^2 + 2\times 7^3 + 6\times 7^4 + 4\times 7^5 + O(7^6)$

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