$\mathcal{L}$-invariants attached to the symmetric square of an elliptic curve

Symmetric square of a newform

- f is a primitive modular form of weight k_0, level N, and trivial character which is ordinary at a prime p (i.e. $a_p(f)$ is a p-adic unit)

$$\boxed{f = \text{primitive form, wieght } k, \text{ level } N, \text{ ord. at } p}$$

- The symmetric square of f is a 3-dimensional ℓ-adic Galois representation

$$\boxed{\mathrm{Sym}^2(\rho_{f,\ell}) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_3(\mathbb{Q}_\ell)}$$

i.e. the symmetric part of $\rho_{f,\ell} \otimes \rho_{f,\ell}$, where $\rho_{f,\ell}$ is the ℓ-adic representation attached to f.

- The (imprimitive) complex L-function attached to f is

$$\boxed{D(s,f) = \prod_q \left(\left(1-\alpha_q^2 q^{-s}\right)\left(1-\beta_q^2 q^{-s}\right)\left(1-\alpha_q\beta_q q^{-s}\right)\right)^{-1}}$$

where

$$\boxed{X^2 - a_q(f)X + q^{k-1} = (X-\alpha_q)(1-\beta_q)}$$

is the Hecke polynomial at q.

- There is a functional equation (Gelbart,Jacquet,Shimura,Li)

$$\boxed{D(s,f) \approx D(2k-1-s,f)}$$

- The critical values are

Theorem (Sturm)**.** *If* $1 \le s \le 2k-2$ *then*

$$\frac{D(s,f,\chi)}{\pi^{(*)}\langle f,f\rangle_N} \in \bar{\mathbb{Q}}$$

where the $^{(*)}$ represents the fact that the critical values naturally separate into two sets $1 \le s \le k-1$ and $k \le s \le 2k-2$ depending on the parity of $\chi(-1)$.

- This allows us to construct p-adic l-functions interpolating twists of $D(s,f)$ by Dirichlet characters of p-power conductor.

Exceptional zeroes

- The interpolation property for the p-adic L-function at $s = k-1$ has the form

$$\boxed{L_p(\mathrm{Sym}^2 f, k-1) = \mathcal{E}_p(k-1) \times \frac{D(k-1,f)}{\pi^{k-1}\langle f,f\rangle_N}}$$

where the term $\mathcal{E}_p(s)$, called the interpolation factor, is necessary for p-adic continuity.

- The interpolation factor vanishes causing the p-adic L-function to vanish at $k-1$ even though the complex L-function does not

$$\boxed{\underbrace{L_p(\mathrm{Sym}^2 f, k-1)}_{=0} = \underbrace{\mathcal{E}_p(k-1)}_{=0} \times \underbrace{\frac{D(k-1,f)}{\pi^{k-1}\langle f,f\rangle_N}}_{\neq 0}}$$

- This motivates the definition of the analytic $\mathcal{L}$-invariant

$$\boxed{\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f) := \mathcal{E}_p^{\dagger}(k-1) \times \frac{L_p'(\mathrm{Sym}^2 f, k-1)}{D(k-1,f)/\pi^{k-1}\langle f,f\rangle_N}}$$

  which relates the first derivative of the p-adic L-function to the algebraic part of the complex L-function.

- Conjecturally, the $\mathcal{L}$-invariant is non-zero

  Conjecture (Coates-Greenberg).
  $$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 f) \neq 0$$

  i.e., the vanishing of the p-adic L-function is due only to the interpolation factor.

- Note, also exceptional zero at $s = k$.

Greenberg's $\mathcal{L}$-invariant

- Greenberg has defined an $\mathcal{L}$-invariant from an algebraic point of view. It is the slope of a certain global cohomology class with respect to a certain basis of the local cohomology.

- For simplicity I will describe it in the case where f arises as the weight 2 form associated with an elliptic curve E.

- The representation here is

$$\boxed{V = \mathrm{Sym}^2(H^1_{\text{ét}}(\bar{E}, \mathbb{Q}_p(1))^*) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2(\mathrm{Ta}_p(E))}$$

- Viewed as a $G_{\mathbb{Q}_p}$ module we have a filtration

$$\boxed{0 = \mathrm{Fil}^3 V \subset \mathrm{Fil}^2 V \subset \mathrm{Fil}^1 V \subset \mathrm{Fil}^0 V = V}$$

  with I_p-isomorphisms ($I_p \subset G_{\mathbb{Q}_p}$)

$$\boxed{\frac{\mathrm{Fil}^i V}{\mathrm{Fil}^{i+1} V} \cong \mathbb{Q}_p(i)}$$

- Consider the Bloch-Kato selmer group

$$\boxed{H^1_{f,\{p\}}(\mathbb{Q}, V) := \mathrm{Ker}\left(H^1(\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}), V) \overset{\oplus \mathrm{res}_\ell}{\to} \bigoplus_{\ell \in \Sigma \setminus \{p\}} H^1(I_\ell, V) \right)}$$

  where

- Σ is the set of primes for which E has bad reduction as well as p
- $\mathbb{Q}_\Sigma$ is the maximum abelian extension of $\mathbb{Q}$ unramified at all primes in Σ

- This is 1-dimensional, so we can fix a generator η, i.e.

$$\boxed{H^1_{f,\{p\}}(\mathbb{Q}, V) = \mathbb{Q}_p \cdot \eta}$$

- Now we explain how to choose co-ordinates of the local cohomolgy.
- First, we know that $H^1(G_{\mathbb{Q}_p}, V) = H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V)$ (both 3-dimensional), so that

$$\boxed{\mathfrak{q} : H^1(G_{\mathbb{Q}_p}, V) = H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V) \overset{\text{mod Fil}^2}{\longrightarrow} H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V / \operatorname{Fil}^2 V)}$$

- Since $\operatorname{Fil}^1 V / \operatorname{Fil}^2 V \cong \mathbb{Q}_p(1)$ as a $G_{\mathbb{Q}_p}$-module, applying Kummer theory

$$\boxed{\mathfrak{q} : H^1(G_{\mathbb{Q}_p}, V) = H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V) \overset{\text{mod Fil}^2}{\longrightarrow} H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V / \operatorname{Fil}^2 V) \overset{\sim}{\longrightarrow} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \left(\varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times\, p^n} \right)}$$

- We have an isomorphism

$$\boxed{\begin{aligned} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \left(\varprojlim_n \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times\, p^n} \right) &\overset{\sim}{\longrightarrow} \mathbb{Q}_p \otimes \mathbb{Q}_p \\ q &\longmapsto (\log_p q, \operatorname{ord}_p q) \end{aligned}}$$

- We map global to local via

$$\boxed{H^1_{f,\{p\}}(\mathbb{Q}, V) \overset{\operatorname{res}_p}{\longrightarrow} H^1(G_{\mathbb{Q}_p}, V)}$$

- Greenberg's $\mathcal{L}$-invariant is the slope of $\mathfrak{q} \cdot \operatorname{res}_p \eta$ inside $H^1(G_{\mathbb{Q}_p}, \operatorname{Fil}^1 V / \operatorname{Fil}^2 V)$

$$\boxed{\mathcal{L}_p^{\text{alg}}(\operatorname{Sym}^2 f) := \frac{\log_p(\mathfrak{q}(\operatorname{res}_p \eta))}{\operatorname{ord}_p(\mathfrak{q}(\operatorname{res}_p \eta))}}$$

Hida theory

- To describe the connection between the analytic and algebraic $\mathcal{L}$-invariants, we turn to Hida theory.
- A Hida family is a formal q-expansion

$$\boxed{F(k) = \sum_{n=1}^{\infty} a_n(k) q^n}$$

where the coefficients are p-adic analytic functions

$$\boxed{a_p(k) : \mathbb{Z}_p \longrightarrow \mathbb{C}_p}$$

that returns classical modular forms when evaluated at integer values of k

$$\boxed{F(k_0) = \text{classical modular form}, k_0 \in \mathbb{Z}^{\geq 2}}$$

- The next theorem connects the analytic and algebraic $\mathcal{L}$-invariants

> **Theorem** (Hida,Harron,Dasgupta,Citro). *If $F(k_0) = f$ then*
> $$\mathcal{L}_p^{\text{alg}}(\text{Sym}^2 f) = -2\frac{a_p'(k_0)}{a_p(k_0)} = \mathcal{L}_p^{\text{an}}(\text{Sym}^2 f)$$

- The algebraic $\mathcal{L}$-invariant generalises naturally to arbitrary modular forms.
- The first equality uses the theory of ordinary representations developed by Hida and Wiles
- The second uses the factorisation of the 3-variable p-adic L-function $L_p(F(k) \otimes G(l), s)$ with $k, l, s \in \mathbb{Z}_p$

$$\begin{aligned} L_p(F(k_0) \otimes F(k_0), k_0 - 1) &= L_p(\text{Sym}^2 F(k_0), k_0 - 1) \times \zeta_p(1) \\ &= L_p(\text{Sym}^2 f, k_0 - 1) \times \zeta_p(1) \\ &= L_p'(\text{Sym}^2 f, k_0 - 1) \times \left(1 - \frac{1}{p}\right) \\ &\approx \mathcal{L}_p^{\text{an}}(\text{Sym}^2 f) \end{aligned}$$

- Similarly, if we study the factorisation of $L_p(F(k_0) \otimes G(l), k_0 - 1)$ we get a factor of $(1 - a_p(l)/a_p(k_0))$.
- So we have that the algebraic and analytic invariants are equal, but what actually are they?

Known results

- If f has complex multiplication then the calculation of the $\mathcal{L}$-invariant reduces to that of the derivative of the Kubota-Leopoldt$L$-function.

$$\text{If } f \text{ has C.M. then } \mathcal{L}_p(\text{Sym}^2 f) = \frac{\log_p \alpha_p^{-2}}{k-1}$$

- A more recent result

> **Theorem** (Rosso, 2016). *If E has split mult. red. at p then*
> $$\mathcal{L}_p(\text{Sym}^2 f_E) = \frac{\log_p(q_E)}{\text{ord}_p(q_E)} \neq 0$$

where q_E is the Tate period of E.

- It is a deep result of transendence thoery that this is non-zero.

Computing $\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E)$ numerically

- We begin with the result

Theorem (Delbourgo, G)**.** *If E = ell. curve conductor $N \leq 300$ with $4 \mid N$ and good ord. red. at $p \leq 13$ then*

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) \neq 0$$

(with a small number of possible exceptions).

- Recalling the definition

$$\mathcal{L}_p^{\mathrm{an}}(\mathrm{Sym}^2 E) := (*) \times \frac{L_p'(\mathrm{Sym}^2 E, 1)}{D(1,E)/\pi\langle f, f\rangle_N}$$

- The quantity $D(1,E)/\pi\langle f, f\rangle_N$ is readily computed so we are then tasked with computing the derivative of the p-adic L-function.

- The p-adic function is constructed via a Mazur-Mellin transform

$$L_p(\mathrm{Sym}^2 E, s) = \int_{x\in\mathbb{Z}_p^\times} \langle x\rangle_p^s d\mu(x)$$

for some measure on $\mathbb{Z}_p$, μ, and so the derivative at $s = 1$ is

$$\begin{aligned} L_p'(\mathrm{Sym}^2 E, 1) &= \int_{x\in\mathbb{Z}_p^\times} \log_p\langle x\rangle_p d\mu(x) \\ &\approx \sum_{e\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \log_p\langle e\rangle_p \mu(e + p^m\mathbb{Z}_p) \end{aligned}$$

i.e. a Riemann sum which can be computed numerically.

- Coates and Schmidt give a formula for the moments

$$\mu(e + p^m\mathbb{Z}_p) \approx \frac{\langle f(z) - \beta_p f(pz), R_{m,e}\rangle_{Np}}{\langle f, f\rangle_N}$$

where the coefficients of $R_{m,e} = \sum r_n(m,e)q^n$ are given explicitly in terms of elementary arithmetic functions.

- We need $4 \mid N$ for this formula to be valid.

- q-expansions of Hecke eigenforms may be computed via modular symbols (implemented in SAGE for example), we can project $R_{m,e}$ onto $f(z)$ and $f(pz)$ and write

$$R_{m,e} = \delta_1 f(z) + \delta_2 f(pz) + \cdots$$

reducing the calculation to the routine exercise of calculating the ratio

$$\frac{\langle f(az), f(bz)\rangle_{Np}}{\langle f, f\rangle_N}$$

for $a, b \in 1, p$.