Iwasawa Main Conjecture over Universal Families (Joint with Oliver Fouquet)

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One of the central problems in number theory is the relation between

$$\begin{aligned} & \text{analytic objects (e.g. special values of } L\text{-functions)} \\ \Leftrightarrow \quad & \text{arithmetic objects.} \end{aligned}$$

- Examples include class number formula for number fields and BSD conjecture for elliptic curves.
- More generally Bloch-Kato (1994) formulated a Tamagawa number conjecture, giving a precise relation between L-functions and (arithmetic) Selmer groups.

We focus on the elliptic modular forms case in this talk. We fix the set up.

- Let p be an odd prime.
- Let f be a cuspidal eigenform of even weight k, trivial character and level N. Write $f = \sum_{n=1}^{\infty} a_n q^n$ with $a_1 = 1$.
- $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathscr{O}_L)$ ($\mathscr{O}_L/\mathbb{Z}_p$ is a finite extension) is the Galois representation associated to f. Let T_f be the representation space. Let $V_f = T_f \otimes \mathbb{Q}_p$.

We suppose the residual representation $\bar{\rho}_f$ is absolutely irreducible.

- Let $\mathbb{Q} \subset \mathbb{Q}_n \subset \mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension with Galois group $\Gamma \simeq \mathbb{Z}_p$. Let $\Lambda = \mathcal{O}_L[[\Gamma]]$ be the Iwasawa algebra.

Definition

On the arithmetic side we define the strict Selmer group as follows.

$$\mathrm{Sel}^{\mathrm{str}}_{\mathbb{Q}_n}(f) := \mathrm{Ker}\{H^1(\mathbb{Q}_n, V_f/T_f(-\frac{k-2}{2})) \to \prod_{v\nmid p} \frac{H^1(\mathbb{Q}_{n,v}, V_f/T_f(-\frac{k-2}{2}))}{H^1_f(\mathbb{Q}_{n,v}, V_f/T_f(-\frac{k-2}{2}))} \times H^1(\mathbb{Q}_{n,p}, V_f/T_f(-\frac{k-2}{2}))\}.$$

The H^1_f is the "finite part" of the local Galois cohomology (we omit the formulas).

Definition

We define

$$\mathrm{Sel}^{\mathrm{str}}_{\mathbb{Q}_\infty}(f) = \varinjlim_n \mathrm{Sel}^{\mathrm{str}}_{\mathbb{Q}_n}(f)$$

and

$$X^{\mathrm{str}}_{\mathbb{Q}_\infty}(f) = \mathrm{Sel}^{\mathrm{str}}_{\mathbb{Q}_\infty}(f)^*$$

where $*$ means Pontryagin dual. It is a finitely generated Λ-module.

Analytic Side

Let Σ be the set of bad primes including p. On the analytic side we define the Iwasawa cohomology by

$$H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma_\infty, T_f(-\frac{k-2}{2})) = \varprojlim_n H^1(\mathbb{Q}^\Sigma_n, T_f(-\frac{k-2}{2}))$$

which is a torsion free module of rank one over Λ.
Kato constructed a "zeta element" $z_{\mathrm{Kato}} \in H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma_\infty, T_f(-\frac{k-2}{2}))$. It is related to special L-values $L(f, \chi, -)$ under the Bloch-Kato dual exponential map, for χ running over characters of Γ.

Conjecture

We have the Iwasawa main conjecture, which states that $X^{\mathrm{str}}_{\mathbb{Q}_\infty}(f)$ is a torsion module over Λ. Moreover we have

$$\mathrm{char}_\Lambda\left(\frac{H^1_{\mathrm{Iw}}(\mathbb{Q}^\Sigma_\infty, T_f(-\frac{k-2}{2}))}{\Lambda z_{\mathrm{Kato}}}\right) = \mathrm{char}_\Lambda(X^{\mathrm{str}}_{\mathbb{Q}_\infty}(f)).$$

Remark

If f is ordinary at p, then the above main conjecture is equivalent to the classical formulation

$$\mathrm{char}(X^{\mathrm{BK}}_{\mathbb{Q}_\infty}) = (\mathscr{L}_{f,p}).$$

In fact Kato constructed a canonical map

$$z_{\mathrm{Kato}} : V_f^{\pm} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_\infty^{\Sigma}, V_f)$$

so that the Iwasawa main conjecture is independent of the choice of the lattice $T_f \subset V_f$.

Kato proved one side divisibility (namely the upper bound for the Selmer group), by constructing an Euler system of zeta element under certain assumptions.

Result at Good Primes

We proved the other side divisibility at good primes.

Theorem

Suppose $p \nmid N$, $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible. Suppose that

- *the image of $\bar{\rho}$ contains a unipotent element;*
- *there is an $\ell | N$ with $\ell \neq p$, such that the ℓ-component of the automorphic representation π_f is of the form $\mathrm{St} \otimes \chi^{\mathrm{ur}}$ with χ^{ur} being the unramfied character sending ℓ to $(-1)\ell^{\frac{k-2}{2}}$.*

Then the Iwasawa main conjecture is true.

Remark

The proof can be summarized as a mixture of Eisenstein congruence and Euler system argument:

- Prove a Greenberg type Iwasawa main conjecture for f twisted by an ordinary CM form of higher weight, using Eisenstein congruences on $\mathrm{U}(3,1)$.
- Use the explicit reciprocity for Beilinson-Flach element studied by Kings-Loeffler-Zerbes to relate the original Iwasawa main conjecture to the Greenberg type above.

Remark

The techniques proving the above theorem do not generalize to cases allowing ramification at p.

- Question: What can we do if $p|N$, especially when $p^2|N$? (The $\pi_{f,p}$ may be supercuspidal in this case, and the usual finite slope deformation theory does not apply.)
- The local Iwasawa theory seems too difficult to study in the ramified cases.
- Answer: One can study the Iwasawa theory along the universal family used in Emerton's local-global compatibility of p-adic Langlands, which contains every classical modular form within it.

Local-Global compatibility and p-adic local Langlands

- For any Galois representation $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathscr{O}_L)$, there is an associated p-adic Banach representation $\pi(\rho_p)$ over $\mathscr{O}_L$.
- More generally for a general coefficient ring R (in application, the universal Galois deformation ring or Hecke algebra) and $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R)$, one can construct a Banach representation $\pi(\rho_p)/R$, using the theory of (φ, Γ)-modules (Colmez).

Completed Cohomology

Definition

Emerton's completed cohomology with coefficient A:
$K^pK_p \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$. Define

$$H^1(K^p)_A := \varinjlim_{K_p} H^1(K_pK^p)_A$$

and

$$\hat{H}^1_{\mathscr{O}} := \varprojlim_{s} H^1(K^p)_{\mathscr{O}}/\varpi^s H^1(K^p)_{\mathscr{O}} = \varprojlim H^1(K^p)_{\mathscr{O}/\varpi^s\mathscr{O}}.$$

Local-Global Compatibility

We fix $\bar{\rho}$ a mod p Galois representation of $G_{\mathbb{Q}}$. Let ρ be its universal deformation over the universal deformation ring R. We fix a finite set Σ of places outside p, and suppose the ℓ-part of K^p is $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for ℓ outside Σ. Then there is a $G_{\mathbb{Q}} \times G_\Sigma \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism

$$\hat{H}^1_{\mathscr{O},\bar{\rho},\Sigma} \simeq \rho \otimes_R \pi_p(\rho^\vee|_{G_{\mathbb{Q}_p}})\hat{\otimes}\pi_\Sigma(\rho),$$

where $G_\Sigma = \prod_{\ell\in\Sigma} \mathrm{GL}_2(\mathbb{Q}_\ell)$ and $\pi_\Sigma(\rho)$ is the local Langlands correspondence of ρ at places in Σ (constructed by Emerton-Helm) over the family R.

Family of Zeta – Nakamura

The following theorem is proved by Nakamura (2020).

Theorem

Let $p \geq 3$, and suppose $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible. Fix a finite set Σ of primes containing all divisors of N prime to p. Suppose the semi-simplification of $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is not of the form $\mathrm{diag}(\omega, 1)$ twisted by a character, and that $\mathrm{End}_{G_{\mathbb{Q}_p}}(\bar{\rho})$ is one dimensional. Then there is a family of zeta elements $z_{\Sigma,n}$, a map

$$\rho \to H^1_{\mathrm{Iw}}(\mathbb{Z}(1/\Sigma_n, \zeta_n), \rho(1)),$$

interpolating the Σ-imprimitive version of Kato's zeta element Euler system at each classical point in $\mathrm{Spec} R$.

Family of Zeta – Colmez-Wang and Zhou

We consider the arc (modular symbol) $\{0,\infty\}\in H_1(K^pK_p)$, which gives a functional

$$\rho\otimes\pi(\rho_p^\vee)\otimes\pi_\Sigma(\rho)\simeq\hat{H}^1_{\mathscr{O},\bar{\rho},\Sigma}\to\mathscr{O}_L.$$

Now for simplicity we ignore the prime to p contribution $\pi_\Sigma(\rho)$. So the arc gives a map

$$f_{\{0,\infty\}}:\rho\to\pi^\vee(\rho_p^\vee).$$

The following are key results for the theory of Colmez-Wang and Zhou.

Theorem

(Colmez-Wang, Zhou) The image of $f_{\{0,\infty\}}$ is invariant under the action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$.

Theorem

(Colmez) There is a canonical isomorphism

$$\pi^\vee(\rho^\vee)^{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} \simeq H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, \rho).$$

Therefore $f_{\{0,\infty\}}$ gives a canonical map

$$\rho \to H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, \rho).$$

We call it z_M (zeta element of modular symbols).

A crucial point proved in Colmez-Wang is that

- The z_M comes from global Iwasawa cohomology $H^1_{\mathrm{Iw}}(\mathbb{Q}^{\Sigma}_{\infty}, \rho)$.

Colmez-Wang also proved the following important results.

- At classical points, the z_M and z_{Kato} are equal.
- The argument of Colmez-Wang is rather complicated and involves global methods.

Euler systems of z_M by fundamental lines

We hope to establish one side divisibility of the Iwasawa main conjecture for z_{Kato} over the universal family R using Euler systems, namely

$$\mathrm{index}(z_{\mathrm{Kato}}) \geq \mathrm{char}(X_{\mathbb{Q}_\infty}^{\mathrm{str}})$$

over R. We need to work with an appropriate context, namely the language of fundamental lines. (It is the Determinant of the Selmer complex with "trivialization" given by zeta element).

It has the following advantages over Characteristic or Fitting ideals of certain Selmer modules.

- It commutes with arbitrary base change.
- It behaves well with respect to changing the finite set of primes Σ.
- It is a principal fraction ideal by definition.

The upper bound for Selmer group in this formulation can thus be reduced to the Euler system argument at each closed point, provided the family is parameterized by a regular ring. If working Selmer modules, one may need a very strong big image assumption which is hard to verify.

Iwasawa Main Conjecture via Fundamental Lines

Definition

(Fundamental Line) Let $Q(A)$ be the fraction field of A. We define the fundamental line

$$\Delta_{\Lambda_3}(\rho) := \mathrm{Det}^{-1}_{\Lambda_3} R\Gamma_{\mathrm{et}}(\mathbb{Z}[1/p], \rho) \otimes_{\Lambda_3} \mathrm{Det}^{-1}_{\Lambda_3} \rho(-1)^{+}.$$

Then there is an isomorphism (trivialization)

$$\mathrm{triv}_{\Lambda_3} : \Delta_{\Lambda_3}(\rho) \otimes_{\Lambda_3} Q(\Lambda_3) \simeq Q(\Lambda_3)$$

induced by the zeta morphism z_M (the cone of the zeta morphism becomes acyclic after tensoring the fraction field, inducing isomorphism on Determinants from the trivial object).

- In sum we prove the descending property of the fundamental line on the non-vanishing locus of the universal zeta elements. (On this locus the zeta morphism is a quasi-isomorphism, and this descending property follows from the functoriality property of determinant functor).
- It is interesting to compare to the recent work of C. H. Kim, J. Lee, G. Ponsinet on the relation between Iwasawa theory of congruent modular forms assuming vanishing of μ-invariant, which amounts to the above descending property at the point corresponding to the maximal ideal of R.

Unfortunately the universal deformation ring R is not regular. Thanks to the following lemma, we can further formulate the problem using regular rings.

Lemma

We can construct a map

$$\Lambda_3 := \mathscr{O}_L[[x_1, x_2, x_3]] \hookrightarrow R$$

which is finite and flat. Moreover we can find a point in $\mathrm{Spec}\mathscr{O}_L[[x_1, x_2, x_3]]$*, such that each point* $y \in \mathrm{Spec}R$ *in the inverse image of* x*, the corresponding Galois representation is crystalline with Hodge-Tate weights in the Fontaine-Laffaille range (i.e.* $k \leq p$*).*

The Iwasawa main conjecture for z_{Kato} can be formulated follows.

Conjecture

The $\mathrm{triv}_{\Lambda_3}$ *sends* $\Delta_{\Lambda_3}(\rho)$ *to* $\Lambda_3 \subset Q(\Lambda_3)$.

Remark

In fact the fundamental line can also be defined over R which behaves well with specializations. However the Euler system argument for family requires working with the regular coefficient ring Λ_3.

Specializations

Remark

By a result of Burns and Flach, the fundamental line for a finite object is trivial. This implies the above Iwasawa main conjecture specialized to $x \in \mathrm{Spec}\Lambda_3$ is the product of the Iwasawa main conjecture for z_M at all points $y \in \mathrm{Spec}R$ in the inverse image of x, as long as the fiber is étale over x. Recall we can find a point x over which the fiber is indeed étale so that either all points corresponds to ordinary forms, or crystalline forms with low (Fontaine-Laffaille) weight.

Now we combine the upper bound result by z_{Kato} over the family and the lower bound at points in the fibre of $\mathrm{Spec}R$ over $x \in \mathrm{Spec}\Lambda_3$ (which are all Fontaine-Laffaille, or all good ordinary). In other words, we have

- $\mathrm{index}(z_{\mathrm{Kato}}) \geq \mathrm{char}(X_{\mathrm{str}})$ or more precisely

$$\mathrm{triv}_{\Lambda_3}(\Delta_{\Lambda_3}(\rho)) \supseteq \Lambda_3$$

  over Λ_3 given by the Euler system argument.

- We have the other side divisibility when specializing to $x \in \mathrm{Spec}\Lambda_3$ at crystalline points (by our first theorem) or at ordinary points (by Skinner-Urban). (Actually we need a slight generalization of those results, namely allowing their twists by certain quadratic character ramified at p, whose proofs are quite straightforward form original results. These points are found via Serre weight conjecture.)

These altogether imply the Iwasawa main conjecture for z_{Kato} over Λ_3.

Theorem

(IMC for z_{Kato} over the universal family) Suppose $p \geq 3$, f is a cuspidal eigenform of even weight k, trivial character and level N. Suppose

- *$\bar{\rho}_f$ is absolutely irreducible;*
- *the semi-simplification of the residual representation $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is not of the form $\mathrm{diag}(\omega, 1)$ twisted by a character;*
- *there is an $\ell | N$ with $\ell \neq p$, such that $\dim_{\mathbb{F}} \bar{\rho}^{I_\ell} = 1$ and $\dim_{\mathbb{F}} \bar{\rho}^{G_\ell} = 0$.*

Then

$$\mathrm{triv}_{\Lambda_3}(\Delta_{\Lambda_3}(\rho)) = \Lambda_3.$$

Iwasawa main conjecture

We can specialize the family version of the Iwasawa main conjecture to a classical point allowing arbitrary ramification, and prove the Iwasawa main conjecture.

Theorem

Assumptions are as in the previous theorem for $\bar{\rho}_f$ for a cuspidal eigenform f. Then the Iwasawa main conjecture for f is true.

Remark

In specializing to classical points, We need a limit argument by etale points of $\mathrm{Spec} R$ over $\mathrm{Spec}\Lambda_3$.

Corollaries

As a corollary of the Iwasawa main conjecture we have the following theorems.

Theorem

(BSD formula when rank is 0 at additive primes) Suppose p is an odd prime and E is an elliptic curve over $\mathbb{Q}$ whose associated weight two eigenform f satisfies the assumption of our main theorem. If the analytic rank r for E is 0, then the p-part of the BSD formula for E is true.

Theorem

Assumptions are as in our main theorem. If $L(f, \frac{k}{2}) = 0$ then the Selmer group for ρ_f has positive rank.

Other Applications

Remark

In fact for more general modular form f we can also deduce a "Tamagawa number conjecture" (generalizing the BSD formula) if $L(f, \frac{k}{2}) \neq 0$, provided we have an appropriate definition for the local Tamagawa number at p.

Remark

There may also be applications to Iwasawa theory for Artin representations (in progress).

Thanks

Thank You !