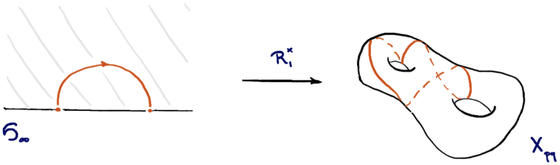
Triple product periods in RM theory, Part II

Iwasawa theory and *p*-adic L-functions, Zhuhai, March 2022
Jan Vonk (Leiden University)



Goal: Twisted triple product periods

These two talks will show how several arithmetic invariants can be viewed as instances of p-adic *twisted triple product periods*

$$\frac{\partial}{\partial s} \left\langle G_s(z,z),\ f_s(z) \right\rangle \Big|_{s=0},$$

where

- $G_s(z_1, z_2)$ is a family of modular forms, of weight $(1,1)$ at $s = 0$.
- $f_s(z)$ is a family of elliptic modular forms, of weight 2 at $s = 0$.

Upshot. Their interpretation as twisted triple products connects these p-adic invariants (which encompass Gross–Stark units, Stark–Heegner points, and RM singular moduli) to p-adic deformations of Artin representations, and makes them approachable / computable. We will discuss computation of p-adic L-functions (joint with A. Lauder).

The resulting analytic formula for Gross–Stark units was recently proved in the spectacular work of Dasgupta–Kakde, in much more generality!

Outline

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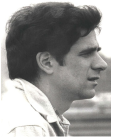
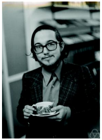
Gross–Zagier (1985)

Let τ_1, τ_2 be two CM points in $\mathcal{H}_\infty = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$, the Poincaré upper half plane. Gross and Zagier (1985) find explicit formula for

$$\mathrm{Nm}\left(j(\tau_1) - j(\tau_2)\right) \quad \in \mathbf{Z}$$

For instance, we have

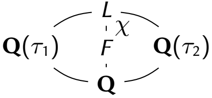
$$\begin{aligned} j\left(\frac{1+\sqrt{-67}}{2}\right) - j\left(\frac{1+\sqrt{-163}}{2}\right) &= -2^{15}\cdot 3^3 \cdot 5^3 \cdot 11^3 + 2^{18}\cdot 3^3\cdot 5^3 \cdot 23^3 \cdot 29^3 \\ &= 2^{15}\cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331 \end{aligned}$$

- **Algebraic proof:** CM elliptic curves, reduces the computation of ord_q to a counting problem in the definite quaternion algebra $B_{\infty q}$.
- **Analytic proof:** Hecke's family of Hilbert Eisenstein series.

Hecke's Eisenstein family

Consider real quadratic field F and genus character χ defined by:



Real analytic Hilbert Eisenstein series $E_s(z_1, z_2)$ defined by:

$$\sum_{[\mathfrak{a}]\in \mathrm{Cl}_F^+} \chi(\mathfrak{a})\, \mathrm{Nm}(\mathfrak{a})^{1+2s} \sideset{}{'}\sum_{(m,n)\,\in\,\mathfrak{a}^2/\mathcal{O}_F^\times} \frac{y_1^s y_2^s}{(mz_1+n)(m'z_2+n')|mz_1+n|^{2s}|m'z_2+n'|^{2s}}$$

One then computes the Fourier expansion of

1. its diagonal restriction $E_s(z, z)$ (vanishes at $s = 0$)
2. its analytic first order derivative with respect to s
3. its holomorphic projection, contained in $M_2(\mathrm{SL}_2(\mathbf{Z})) = \{0\}$.

The first Fourier coefficient is of the form

$$\log \mathrm{Nm}(j(\tau_1) - j(\tau_2)) \; + \; \sum_q \mathrm{Int}_q \cdot \log(q)$$

Towards an RM theory?

(With Darmon) A real quadratic $K_1 \hookrightarrow B := M_2(\mathbf{Q})$ defines a class in

$$\mathrm{H}^1\left(\mathrm{SL}_2(\mathbf{Z}[1/p]),\ \mathcal{M}^\times / \mathbf{C}_p^\times\right),$$

which is "evaluated" at $K_2 \hookrightarrow B$. Here $\Gamma := \mathrm{SL}_2(\mathbf{Z}[1/p])$ acts on

$$\begin{array}{rcl} \mathcal{A} & = & \text{Holo. functions on } \mathcal{H}_p \\ \mathcal{M} & = & \text{Mero. functions on } \mathcal{H}_p \end{array}$$

with $\mathcal{H}_p$ the p-adic upper half plane.

- When p is **split** in K_1, the cocycle is *analytic*, i.e. valued in $\mathcal{A}^\times / \mathbf{C}_p^\times$ and there are two types of invariants
  - Gross–Stark units
  - Stark–Heegner points
- When p is **non-split** in K_1, get RM singular moduli.

Triple product periods (joint with Darmon and Pozzi)

For a pair of embeddings of

$$\begin{aligned} K_1 &= \mathbf{Q} \times \mathbf{Q}, \\ K_2 &= \text{Real quadratic with } p \text{ inert}, \end{aligned}$$

into the quaternion algebra $B = M_2(\mathbf{Q})$ we obtain a weight $(1, 1)$ form over $F \simeq K_2$ is associated to an odd unramified character ψ, with has Fourier expansion at the cusp $\mathfrak{d}$ given by:

$$E_{\psi}^{(p)}(z_1, z_2) := L_p(\psi, 0) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \left(\sum_{p \nmid I \,|\, (\nu)\, \mathfrak{d}} \psi(I) \right) e^{2\pi i(\nu_1 z_1 + \nu_2 z_2)}.$$

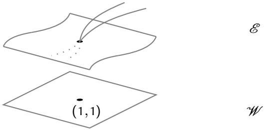
Q: *Find p-adic family $G_s(z_1, z_2)$ of Hilbert modular forms such that*

$$G_0(z_1, z_2) = E_{\psi}^{(p)}(z_1, z_2).$$

Triple product periods (joint with Darmon and Pozzi)

All possible families of *eigenforms* are encoded in a geometric object: The *eigenvariety* $\mathscr{E}$, with a natural map $\pi : \mathscr{E} \to \mathscr{W}$ to weight space.

The eigenvariety around the point of weight $(1, 1)$ defined by the Eisenstein series $E_{\psi}^{(p)}$ was described by Betina–Dimitrov–Shih:

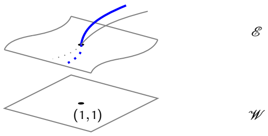


Focus: (1) The Eisenstein family in parallel weight $(1 + s, 1 + s)$
(2) The cuspidal family in anti-parallel weight $(1 + s, 1 - s)$.

[DPV1] *Diagonal restrictions of p-adic Eisenstein families*
[DPV2] *On the RM values of the Dedekind–Rademacher cocycle*

Outline

Triple product periods (DPV1)



We can consider the Eisenstein family in parallel weight:

$$G_s(z_1, z_2) := L_p(\psi, s) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \left(\sum_{p \nmid I \mid (\nu)\,\mathfrak{d}} \psi(I)\,\mathrm{Nm}(I)^s \right) e^{2\pi i(\nu_1 z_1 + \nu_2 z_2)},$$

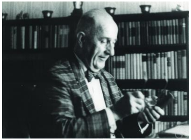
Its *diagonal restriction* is obtained by setting $z_1 = z_2$, a modular form of weight $2(1+s)$ and level $\Gamma_0(p)$. There is a dichotomy between constant term (L-value) and higher Fourier coefficients (elementary divisor sums).

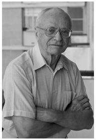
Rationality of L-values

Siegel used the diagonal restriction to show the rationality of $L(\psi, 1-k)$, and to *compute* some of its special values. For instance:

$$\zeta_F(-1) = \frac{-1}{60} \sum_{\substack{b<\sqrt{D} \\ b\equiv D \pmod 2}} \sigma_1\left(\frac{D-b^2}{4}\right)$$

This idea goes back to Hecke, and was used by Klingen and Siegel to show the rationality of L-values of totally real fields at negative integers. Serre used it to construct certain p-adic L-functions.





Rationality of L-values: Example

Consider $F = \mathbf{Q}(\sqrt{5})$. The narrow ray class group of conductor (3) is

$$\mathrm{Cl}^+_{(3)} \simeq \mathbf{Z}/2\mathbf{Z}$$

and the unique quadratic character ψ of conductor (3) is totally odd. Compute the diagonal restriction for $k = 3$:

$$L(\psi, -2) \; - 1144q - 39696q^2 - 291448q^3 - 1261696q^4 + \dots$$

This is a modular form of weight 6 and level $\Gamma_1(3)$, and the space $M_6(\Gamma_1(3))$ is 3-dimensional, and has a basis of the form

$$\begin{cases} f_1 = 1 \quad - 504q^3 + \dots, \\ f_2 = q + 45q^3 + 166q^4 + \dots, \\ f_3 = q^2 + 6q^3 + 27q^4 + \dots \end{cases}$$

We determine that the diagonal restriction is equal to

$$\frac{32}{9} f_1 - 1144 f_2 - 39696 f_3 \quad \Rightarrow L(\psi, -2) = 32/9.$$

Computing p-adic L-functions

The computation of the constant term can be reduced to the computation of the higher Fourier coefficients (note: independence of weights!)

$$a_n = 4 \sum_{\substack{\nu \in \mathfrak{d}_+^{-1} \\ \mathrm{Tr}(\nu)=n}} \sum_{\substack{I|(\nu)\,\mathfrak{d} \\ (I,p)=1}} \psi(I) \langle \mathrm{Nm}(I) \rangle^{k-1}$$

For instance, when ψ is unramified odd character of $F = \mathbf{Q}(\sqrt{141})$, and $p = 5$:

$$\begin{aligned} L_p(\psi, T) \quad = \quad & -32552075 + O(5^{11}) - (4812777 + O(5^{10}))T - (1093284 + O(5^{10}))T^2 \\ & -(4814847 + O(5^{10}))T^3 + O(T^4), \qquad \text{where } T = (1+p)^s - 1. \end{aligned}$$

We see that this 5-adic L-function has a unique zero (at $T = 1992099 \cdot 5 \bmod 5^{10}$). Note that the field $L = F(\psi) = \mathbf{Q}(\sqrt{141}, \sqrt{-3})$ has class group $\mathrm{Cl}(L) = \mathbf{Z}/5\mathbf{Z}$.

λ-invariants

The λ and μ invariants are defined by the factorisation

$$L_p(\psi, T) = p^{\mu} P(T) U(T), \qquad U(T) \in \mathbf{Z}_p[\![T]\!]^{\times}$$

where $P(T)$ is a distinguished polynomial, of degree λ.

Q: *How does λ vary as ψ runs over quadratic characters of F?*

Ellenberg–Jain–Venkatesh make random matrix model for this, they predict that the proportion of $\lambda = r$ is given by

$$p^{-r} \prod_{i>r} (1 - p^{-i})$$

Model by eigenvalues "close to 1" of a random matrix. Same statistics for the groups $\mathrm{GL}(\mathbf{Z}_p)$ and $\mathrm{GSp}(\mathbf{Z}_p)$! Computations for $\approx 10^6$ discriminants:

λ	0	1	2	3	4	5	6
Predicted	0.589887	0.2680	0.0936	0.0322	0.0109	0.0035	0.0012
Observed	0.5601	0.2800	0.1050	0.0363	0.0122	0.0041	0.0013

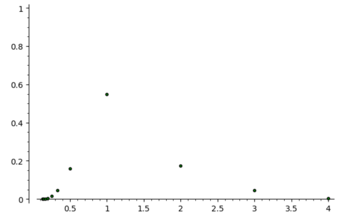
λ-invariants

Q: *Which matrix group models the zeroes of p-adic L-functions?*

To decide between $\mathrm{GL}(\mathbf{Z}_p)$ or $\mathrm{GSp}(\mathbf{Z}_p)$, can look at the statistics of the *smallest* zero. Exclude the exceptional zeroes arising from

$$L_p(\psi, 0) = (1 - \psi(p)p^0) \cdot L(\psi, 0)$$

Giacomo Santato observes p-adic Hazelgrave phenomenon (see plot). This is mirrored by the statistics of the eigenvalues of $\mathrm{GSp}(\mathbf{Z}_p)$.



Twisted triple product periods

Important fact: The *diagonal restriction* of the Eisenstein family $G_s(z,z)$ vanishes at $s=0$, i.e. $G_s(z,z)=0$. This implies that the first derivative with respect to s is an *overconvergent* modular form of weight 2.

- The triple product period with the Eisenstein series E_2 is given by

$$\left\langle \frac{\partial}{\partial s} G_s(z,z)\Big|_{s=0},\ E_2 \right\rangle = \log_p \left(\mathrm{Nm}_{\mathbf{Q}_p}\, \Theta_{\mathrm{DR}}[\psi]\right) \overset{?}{=} \log_p \left(\mathrm{Nm}_{\mathbf{Q}_p}\, u_\psi\right)$$

  where $u_\psi \in (\mathbf{Q} \otimes \mathcal{O}_H[1/p]^\times)^\psi$ is the Gross–Stark unit.
- The triple product period with a cusp form f is given by

$$\left\langle \frac{\partial}{\partial s} G_s(z,z)\Big|_{s=0},\ f \right\rangle = \log_p \left(\mathrm{Nm}_{\mathbf{Q}_p}\, \Theta_f[\psi]\right) \overset{?}{=} \log_E \left(\mathrm{Nm}_{\mathbf{Q}_p}\, P_\psi\right)$$

  where $P_\psi \in (\mathbf{Q} \otimes E(H))^\psi$ is a Stark–Heegner point.

Example

Let $F = \mathbf{Q}(\sqrt{21})$ and $p = 11$, then the space of weight 2 forms on $\Gamma_0(p)$ is spanned by E_2, and the form f associated to

$$E = X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20.$$

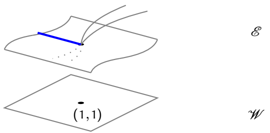
Take ψ associated to $F(\sqrt{-3})$. Using algorithms of Lauder, we find

$$\begin{aligned} \left\langle \tfrac{\partial}{\partial s} G_s(z,z)\Big|_{s=0},\ E_2 \right\rangle &= \tfrac{16}{5} \log_{11}\left(\tfrac{2+\sqrt{-7}}{11}\right) \\ \left\langle \tfrac{\partial}{\partial s} G_s(z,z)\Big|_{s=0},\ f \right\rangle &= \tfrac{8}{5} \log_E\left(\tfrac{1-\sqrt{-7}}{2}, 1+2\sqrt{-7}\right). \end{aligned}$$

This computation uses techniques of Lauder for computing with spaces of overconvergent modular forms and Hida's ordinary projector.

Outline

Triple product periods (DPV2)



Alternatively, we can consider the cuspidal family $G_s(z_1, z_2)$ in anti-parallel weight. This family is much more subtle!

It is **not explicit**, and its Fourier coefficients need to be calculated first, using the p-adic deformation theory of the Galois representation

$$\rho = 1 \oplus \psi \; : \; \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p).$$

Analytic family

Strategy: Can compute anti-parallel cuspidal family from the corresponding Galois deformation, takes place in

$$\mathrm{H}^1(G_F, \mathrm{Ad}(\rho)) \simeq \mathrm{H}^1(G_F, \overline{\mathbf{Q}}_p) \oplus \mathrm{H}^1(G_F, \psi) \oplus \mathrm{H}^1(G_F, \psi^{-1}) \oplus \mathrm{H}^1(G_F, \overline{\mathbf{Q}}_p)$$

Let H be the field cut out by the character ψ. By inflation-restriction + global class field theory we obtain the isomorphisms

$$\begin{aligned} \mathrm{H}^1(G_F, \psi) &\simeq \mathrm{Hom}(G_H, \overline{\mathbf{Q}}_p)[\psi] \\ &\simeq \mathrm{Hom}((\mathcal{O}_H \otimes \mathbf{Z}_p)^\times, \overline{\mathbf{Q}}_p)[\psi] \end{aligned}$$

Evaluation gives Gross–Stark units u_ψ in p-unit group $\mathcal{O}_H[1/p]^\times$.

Q: *Which deformations are modular? In the anti-parallel direction?*

Analytic family

Define (representable) deformation problem $\mathcal{D}_{\rho,L}$ that captures properties of the Galois representation attached to the whole cuspidal family.

Some important features are

- deformations ρ_A should be *nearly ordinary*, i.e.

$$\rho_A \big|_{D_p} \sim \begin{pmatrix} \lambda_p & * \\ 0 & \mu_p \end{pmatrix}$$

  for some characters $\lambda_\mathfrak{p}, \mu_\mathfrak{p} : D_\mathfrak{p} \to A$
- The G_{F_p}-stable rank one summand L is part of data, to rigidify.

It is representable by a deformation ring $R_{\rho,L}$, and can show there is a natural isomorphism with the relevant Hecke algebra.

Finally, we compute the tangent space to $R_{\rho,L}$ in $\mathrm{H}^1(G_F, \mathrm{Ad}(\rho))$ (6-dim) with nearly ordinary (codim 3) and anti-parallel (codim 2) conditions.

Analytic family

The unique **cuspidal** deformation in *anti-parallel* weight is computed from deformations of $1 \oplus \psi$, related to Gross–Stark units $\log_p(u_\psi)$.
The two **Eisenstein** deformations in *parallel* weight are explicit.

We define a combination $G_s(z_1, z_2)$ of all three families. Then, we consider

1. its diagonal restriction $G_s(z, z)$ vanishes at $s = 0$
2. its analytic first order derivative with respect to s
3. its *ordinary* projection, contained in $M_2(\Gamma_0(p))$

has first Fourier coefficient of the form

$$\log_p \Theta_{\mathrm{wind}}[\psi] + \log(u_\psi), \qquad \text{where } u_p \in \mathbf{Q} \otimes \mathcal{O}_H[1/p]^\times$$

Remark. The first order derivative of the family $G_s(z_1, z_2)$ is **not** a modular form. Its diagonal restriction however is a p-adic overconvergent form.

Example

Let $F = \mathbf{Q}(\sqrt{136})$ with $\mathrm{Cl}^+(F) \simeq \mathbf{Z}/4\mathbf{Z}$, and $p = 19$ inert in F. Computing the higher Fourier coefficients, and writing them as a linear combination of elements of the Katz basis for 19/20-overconvergent forms yields the constant terms $\log_{19}(u)$ computed to precision 19^{50}, in less than 3 seconds. Recognised as a root of the polynomial

$$361x^4 + 508x^3 + 310x^2 + 508x + 361 = 0,$$

which generates the narrow Hilbert class field over F.

Conclusion. The interpretation of Gross–Stark units and Stark–Heegner points as twisted triple products connects these p-adic invariants to the p-adic deformation theory of Artin representations, and makes them both theoretically and computationally approachable.