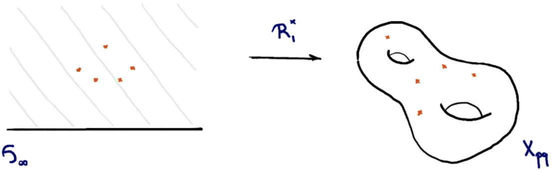
Triple product periods in RM theory, Part I

Iwasawa theory and *p*-adic L-functions, Zhuhai, March 2022

Jan Vonk (Leiden University)



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These two talks will show how several arithmetic invariants can be viewed as instances of p-adic *twisted triple product periods*

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Upshot. Their interpretation as twisted triple products connects these p-adic invariants to the p-adic deformation theory of Artin representations, and makes them approachable / computable. We will discuss applications to computing p-adic L-functions (joint with A. Lauder).

Outline

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1. CM Theory: Classical era
2. CM Theory: Gross–Zagier
3. Towards an RM Theory?

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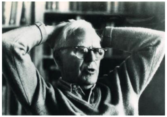

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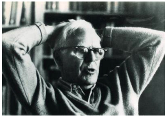

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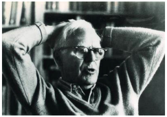

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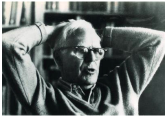

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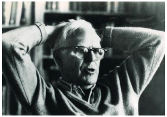

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Explanation comes from the theory of *complex multiplication*.

Singular moduli I: Classical era

Consider Klein's modular j-function

$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots \qquad q = e^{2\pi iz}$$

This function satisfies

$$j\left(\frac{az+b}{cz+d}\right) = j(z), \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

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The values of this function at $z \in K$ quadratic imaginary are called *singular moduli*. They are always algebraic integers, e.g

$$\begin{aligned} j(\sqrt{-1}) &= 1728 \\ j(\sqrt{-5}) &= 2^6 \cdot 5 \cdot (884\sqrt{5} + 1975) \\ j(\sqrt{-14}) &= 2^3 \left(323 + 228\sqrt{2} + (231 + 161\sqrt{2})\sqrt{2\sqrt{2} - 1}\right)^3 \end{aligned}$$

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Initial interest centered around their role in *explicit class field theory*.

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Field	$E_{\mathbf{Q}}$ with CM by order of τ	$j(\tau)$
$\mathbf{Q}(\sqrt{-1})$	$y^2 = x^3 + x$	$2^6 \cdot 3^3$
$\mathbf{Q}(\sqrt{-2})$	$y^2 = x^3 + x$	$2^6 \cdot 5^3$
$\mathbf{Q}(\sqrt{-3})$	$y^2 + xy = x^3 - x^2 - 2x - 1$	0
$\mathbf{Q}(\sqrt{-7})$	$y^2 = x^3 + 4x^2 + 2x$	$-3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-11})$	$y^2 + y = x^3 - x^2 - 7x + 10$	-2^{15}
$\mathbf{Q}(\sqrt{-19})$	$y^2 + y = x^3 - 38x + 90$	$-2^{15} \cdot 3^3$
$\mathbf{Q}(\sqrt{-43})$	$y^2 + y = x^3 - 860x + 9707$	$-2^{18} \cdot 3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-67})$	$y^2 + y = x^3 - 7370x + 243528$	$-2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3$
$\mathbf{Q}(\sqrt{-163})$	$y^2 + y = x^3 - 2174420x + 1234136692$	$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$
$\mathbf{Q}(\sqrt{-1})$	$y^2 = x^3 - 11x + 14$	$2^3 \cdot 3^3 \cdot 11^3$
$\mathbf{Q}(\sqrt{-3})$	$y^2 = x^3 - 15x + 22$	$2^4 \cdot 3^3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-3})$	$y^2 + y = x^3 - 30x + 63$	$-2^{15} \cdot 3 \cdot 5^3$
$\mathbf{Q}(\sqrt{-7})$	$y^2 = x^3 - 595x + 5586$	$3^3 \cdot 5^3 \cdot 17^3$

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This explains the observation on our first slide!

$$-2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3 = j\left(\frac{1+\sqrt{-163}}{2}\right) = -e^{\pi\sqrt{163}} + 744 + (\text{very small}).$$

The Birch–Swinnerton-Dyer conjecture

After World War II, a renaissance of CM theory took place following the efforts to investigate the Birch–Swinnerton-Dyer conjecture. This conjecture predicts a connection between *algebraic* and *analytic* invariants of an elliptic curve. For $E/\mathbf{Q}$ such that $E(\mathbf{Q})$ is of rank r, it asserts that

$$L^{(r)}(E,1) = r! \cdot \frac{\Omega_E \cdot \mathrm{Reg}_E \cdot |\text{Ш}_E| \cdot \prod_v c_v}{|E(\mathbf{Q})_{\mathrm{tor}}|^2}$$

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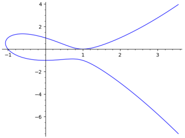
Let us explore it in the example

$$E : y^2 + xy = x^3 - x^2 - x + 1$$

which has conductor 58 and j-invariant $-3^3 \cdot 19^3/2^2 \cdot 29$.



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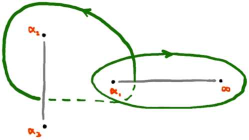
- **Periods:** Have $E \simeq \mathbf{C}/\Lambda$ where Λ is the period lattice

$$\begin{aligned}
\Lambda &= \left\{ \int_\gamma \frac{dx}{2y+x} \ : \ \gamma \in \mathrm{H}_1(E(\mathbf{C}), \mathbf{Z}) \right\} \\
&= \mathbf{Z} \cdot \int_{\alpha_1}^{\infty} \frac{2dx}{\sqrt{4x^3 - 3x^2 - 4x + 4}} + \mathbf{Z} \cdot \int_{\alpha_1}^{\alpha_2} \frac{2dx}{\sqrt{4x^3 - 3x^2 - 4x + 4}} \\
&= \mathbf{Z} \cdot (5.4656\ldots) + \mathbf{Z} \cdot (2.7328\ldots + i1.1118\ldots)
\end{aligned}$$



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- **Mordell–Weil:** We have $E(\mathbf{Q}) \simeq \mathbf{Z} = \langle P_0 \rangle$ where $P_0 = (0, 1)$. Its canonical height is easily computed to be

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- **Tamagawa:** For every place v of $\mathbf{Q}$, compute $c_v := |E(\mathbf{Q}_v)/E^0(\mathbf{Q}_v)|$.
  - $v = \infty$: Unique real component, $c_v = 1$.
  - $v = 2$ find $c_v = 2$, and $v = 19$ find $c_v = 1$.

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such that

$$\begin{cases} Y^2 + XY & = & X^3 - X^2 - X + 1, \\ dX/(2Y+X) & = & 2\pi i f(\tau) d\tau, \end{cases}$$

where $f(\tau) = q + a_2 q^2 + a_3 q^3 + \ldots$ with $q = e^{2\pi i \tau}$, a cusp form of weight two with Fourier coefficients $a_p = p + 1 - |E(\mathbf{F}_p)|$.

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$$\begin{aligned} X(\tau) &= q^{-2} + q^{-1} + 3 + 3q + 7q^2 + 7q^3 + 14q^4 + \ldots \\ Y(\tau) &= -q^{-3} - 2q^{-2} - 5q^{-1} - 8 - 16q - 24q^2 - 44q^3 + \ldots \end{aligned}$$

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$$L(E,s) := \sum_{n\geq 1} \frac{a_n}{n^s}, \qquad \mathrm{Re}(s) > 3/2$$

analytically continues to $s \in \mathbf{C}$ and satisfies

$$\begin{aligned} \Lambda(E,s) &:= 58^{s/2} \cdot (2\pi)^{-1} \cdot \Gamma(s) \cdot L(E,s) \\ &= \int_0^\infty f\left(\frac{it}{\sqrt{58}}\right) t^s \frac{dt}{t} \\ &= -\Lambda(E, 2-s). \end{aligned}$$

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This implies that we have

$$\begin{aligned} L(E,1) &= 0 \\ L'(E,1) &\approx 0.46370\ldots \approx \frac{\Omega_E \cdot \widehat{h}(P_0) \cdot |Ш_E| \cdot \prod_v c_v}{|E(\mathbf{Q})_{\mathrm{tor}}|^2} \end{aligned}$$

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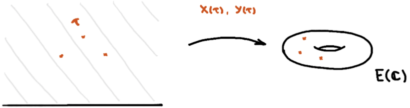

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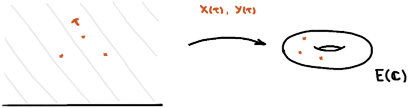

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$$Q_D := \left\{[a, b, c] : b^2 - 4ac = D,\ 58 \,|\, a > 0\right\}.$$

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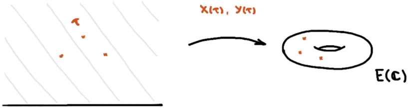

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$$Q_D := \left\{[a, b, c] : b^2 - 4ac = D,\ 58 \mid a > 0\right\}.$$

For any element in Q_D the image of $\tau := \frac{-b+\sqrt{D}}{2a}$ on E is algebraic, defined over the Hilbert class field of $Q(\sqrt{D})$. The number of Γ-orbits on Q_D is

$$|Q_D / \Gamma| = \#\mathrm{Cl}(\mathbf{Q}(\sqrt{D}))$$

and the images of the corresponding points on $E(\mathbf{C})$ are a full Galois orbit. In particular, their sum defines a point in $E(\mathbf{Q})$.

Heegner points

Example 1. When $D = -23$ we find $[116, 21, 1] \in Q_{-23}$ and

$$\tau = \frac{-21 + \sqrt{-23}}{232} \mapsto (0.8774\ldots - i0.7448\ldots, -0.2150\ldots + i1.3071\ldots)$$

defined over $\mathbf{Q}(\alpha)$ where $\alpha^3 - \alpha^2 + 1 = 0$. There are 3 orbits on Q_{-23} for the action of Γ, and the sum of the corresponding points is

$$\begin{aligned}(0.8774\ldots - i0.7448\ldots, -0.215\ldots + i1.3071\ldots) &\quad + \\ (0.8774\ldots + i0.7448\ldots, -0.215\ldots - i1.3071\ldots) &\quad + \\ (-0.7548\ldots, -0.5698\ldots) &\quad = \quad (1, 0) \;=\; -2P_0\end{aligned}$$

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Example 2. When $D = -71$ we find $[174, 25, 1] \in Q_{-71}$ and

$$\tau = \frac{-25 + \sqrt{-71}}{348} \mapsto (-0.3448\ldots - i1.0787\ldots, 2.0250\ldots + i0.9148\ldots)$$

which is defined over $\mathbf{Q}(\alpha)$ with $\alpha^7 + \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha^2 + 2\alpha - 1 = 0$. There are 7 orbits for Γ, the sum of the corresponding points is 0.

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Let us make a more systematic computation. Let us denote

$$P_d = b_d P_0, \qquad b_d \in \mathbf{Z}$$

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Gross–Kohnen–Zagier show that this is the q-expansion of a modular form of weight $3/2$, which is attached to the elliptic curve E under the Shimura correspondence.

Proof: Make detailed study of *height pairings* of Heegner points.

Outline

Gross–Zagier (1985)

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- **Algebraic proof:** CM elliptic curves, reduces the computation of ord_q to a counting problem in the definite quaternion algebra $B_{\infty q}$.
- **Analytic proof:** Hecke's family of Hilbert Eisenstein series.

1. Algebraic proof

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In above example, for $q = 3$, have $B_{3\infty} = \langle 1, x, y, z \rangle$ with $x^2 = z^2 = -3, y^2 = -1$. We get arithmetic intersection number

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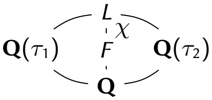
Similarly compute $\mathrm{Int}_2(\alpha_1, \alpha_2) = 9 \times 1 + 2 + 2 + 2 = 15$, etc.

2. Analytic proof

Consider real quadratic field F and genus character χ defined by:



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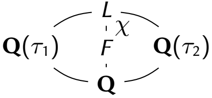

Real analytic Hilbert Eisenstein series $E_s(z_1, z_2)$ defined by:

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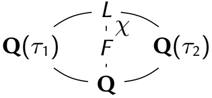

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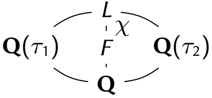

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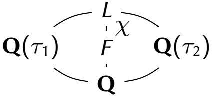

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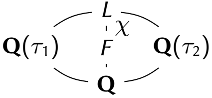

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The first Fourier coefficient is of the form

$$\log \mathrm{Nm}(j(\tau_1) - j(\tau_2)) \;+\; \sum_q \mathrm{Int}_q \cdot \log(q)$$

Theta lifts

Renaissance after Gross–Zagier saw tremendous developments in the works of Kudla, Yuan–Zhang–Zhang, Howard–Yang, etc.

Given embeddings of quadratic fields $K_1, K_2 \hookrightarrow B$ into $B = M_2(\mathbf{Q})$. Consider the quadratic space $V = B \times B$ with quadratic form

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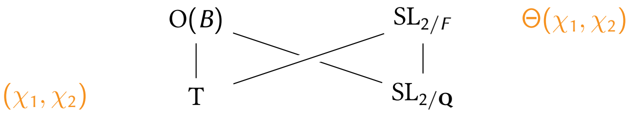

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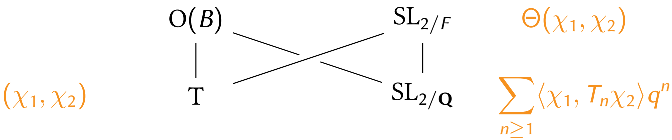

Outline

Triple product periods (joint with Darmon and Pozzi)

For a pair of embeddings of

$$\begin{aligned} K_1 &= \mathbf{Q} \times \mathbf{Q}, \\ K_2 &= \text{Real quadratic with } p \text{ inert}, \end{aligned}$$

into the quaternion algebra $B = M_2(\mathbf{Q})$ we obtain a weight $(1, 1)$ form over $F \simeq K_2$ is associated to an odd unramified character ψ, with has Fourier expansion at the cusp $\mathfrak{d}$ given by:

$$E_{\psi}^{(p)}(z_1, z_2) := L_p(\psi, 0) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \left(\sum_{p \nmid I \,|\, (\nu)\, \mathfrak{d}} \psi(I) \right) e^{2\pi i(\nu_1 z_1 + \nu_2 z_2)}.$$

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Q: *Find p-adic family $G_s(z_1,z_2)$ of Hilbert modular forms such that*

$$G_0(z_1,z_2) = E_{\psi}^{(p)}(z_1,z_2).$$

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All possible families of *eigenforms* are encoded in a geometric object: The *eigenvariety* $\mathscr{E}$, with a natural map $\pi : \mathscr{E} \to \mathscr{W}$ to weight space.

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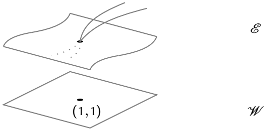

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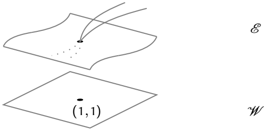
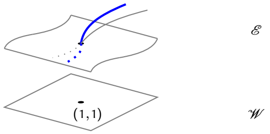


Focus: (1) The Eisenstein family in parallel weight $(1 + s, 1 + s)$
(2) The cuspidal family in anti-parallel weight $(1 + s, 1 - s)$.

[DPV1] *Diagonal restrictions of p-adic Eisenstein families*
[DPV2] *On the RM values of the Dedekind–Rademacher cocycle*

Triple product periods (DPV1)



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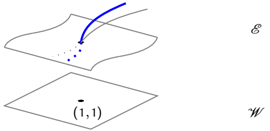

We can consider the Eisenstein family in parallel weight:

$$G_s(z_1, z_2) := L_p(\psi, s) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \left(\sum_{p \nmid I \mid (\nu)\,\mathfrak{d}} \psi(I)\,\mathrm{Nm}(I)^s \right) e^{2\pi i(\nu_1 z_1 + \nu_2 z_2)},$$

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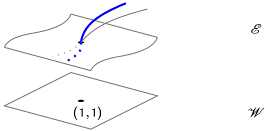

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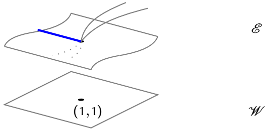
It has the features that

- It is **completely explicit**, can be used to compute p-adic L-functions.
- By the Gross–Stark conjecture, we get twisted triple product

$$\frac{\partial}{\partial s} \left\langle G_s(z, z), E_2(z) \right\rangle \Big|_{s=0} = L_p'(\psi, 0) = \log_p(\mathrm{Nm}\, u_\psi).$$

where u_ψ is a Gross–Stark unit in $\mathbf{Q} \otimes \mathcal{O}_H[1/p]^\times$.

Triple product periods (DPV2)



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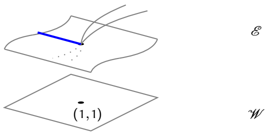

Alternatively, we can consider the cuspidal family $G_s(z_1, z_2)$ in anti-parallel weight. This family is much more subtle!

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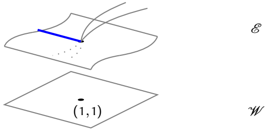

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- It is **not explicit**, and its Fourier coefficients need to be calculated first, using the p-adic deformation theory of the Galois representation

$$\rho = 1 \oplus \psi \ : \ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p).$$

- We get a more refined twisted triple product

$$\frac{\partial}{\partial s} \left\langle G_s(z,z), E_2(z) \right\rangle \Big|_{s=0} = \log_p(u_\psi).$$